Q2 : (40 Marks)

A) For $F(s)=\dfrac{(2s-7)}{(s^2+25)}$, it is required to find:-

- It's related Controllable state space representation matrices. (5 Marks)

$F(s)=\dfrac{2s-7}{s^2+25}$ then $A=\begin{bmatrix}0 & 1\\ -25 & 0\end{bmatrix}, B=\begin{bmatrix}0\\1\end{bmatrix}, C=\begin{bmatrix}-7 & 2\end{bmatrix}, D=0$

- It's related diagonal state-space representation matrices. (10 Marks)

$\dfrac{2s-7}{s^2+25}=\dfrac{2s-7}{(s-5j)(s+5j)}=\dfrac{A}{(s-5j)}+\dfrac{B}{(s+5j)}$ using Partial Fraction Method

$2s-7=A(s+5j)+B(s-5j)$, solve for $A=1+0.7j$, $B=1-0.7j$

thus; $\dfrac{2s-7}{s^2+25}=\dfrac{1+0.7j}{(s-5j)}+\dfrac{1-0.7j}{(s+5j)}$ then the diagonal s.s. Representation

matricies are: $A=\begin{bmatrix}5j & 0\\ 0 & -5j\end{bmatrix}, B=\begin{bmatrix}1\\1\end{bmatrix}, C=\begin{bmatrix}1+0.7j & 1-0.7j\end{bmatrix}, D=0$

- f(t) using Laplace inverse properties. (5 Marks)

$\mathcal{L}^{-1}\left(\dfrac{2s-7}{s^2+25}\right)=2\mathcal{L}^{-1}\left(\dfrac{s}{s^2+25}\right)-\dfrac{7}{5}\mathcal{L}^{-1}\left(\dfrac{5}{s^2+25}\right)=2\cos(5t)-\dfrac{7}{5}\sin(5t).$

B) solve $\ddot{y}+9y=18t$ with $y(0)=0$, $\dot{y}(0)=6$. (10 Marks)

$\mathcal{L}y''=s^2y(s)-sy(0)-y'(0)=s^2y(s)-6$, $\mathcal{L}\,18t=18/s^2$

thus; $s^2y(s)+9y(s)=6+\dfrac{18}{s^2}\Rightarrow y(s)=\dfrac{6}{(s^2+9)}+\dfrac{18}{s^2(s^2+9)}$

$\dfrac{18}{s^2(s^2+9)}=\dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{Cs+D}{(s^2+9)}$ P.F.M gives $A=0, B=2, C=0, D=-2$

thus $\dfrac{18}{s^2(s^2+9)}=\dfrac{2}{s^2}+\dfrac{-2}{(s^2+9)}\Rightarrow y(s)=2/s^2+\dfrac{4}{(s^2+9)}$ then $\mathcal{L}^{-1}y(s)=y(t)=2t+\frac{4}{3}\sin(3t)$.

C) Given $y''-6y=g(t)$, with $y(0)=y'(0)=0$, and $g(t)=\begin{cases}0 & t<\pi\\ \sin(t-\pi) & t\geq\pi\end{cases}$

it is required to show y(s) in factorized form. (10 Marks)

$\mathcal{L}[y''-6y]=\mathcal{L}g(t)\Rightarrow(s^2-6)y(s)=\dfrac{e^{-\pi s}}{(s^2+1)}\Rightarrow\therefore y(s)=\dfrac{e^{-\pi s}}{(s^2+1)(s^2-6)}.$

$\dfrac{1}{(s^2+1)(s^2-6)}=\dfrac{As+B}{(s^2+1)}+\dfrac{Cs+D}{(s^2-6)}\Rightarrow$ P.F. gives $A=C=0, B=-1/7, D=1/7$

thus $y(s)=\frac{1}{7}e^{-\pi s}\left[\dfrac{-1}{(s^2+1)}+\dfrac{1}{(s^2-6)}\right].$

Q3: (40 Marks)

A) Given $\ddot{y}+2\dot{y}+y=x$. It is required to find matrix exponential (e^{At}) using Cayley_Hamilton theorem. (10 Marks).

$$\frac{y}{x}=\frac{1}{s^2+2s+1} \Rightarrow A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix} \Rightarrow |\lambda I-A|=0 \text{ gives } \lambda_1=\lambda_2=-1$$

$$f(\lambda_1)=g(\lambda_1)=\alpha_0+\alpha_1\lambda_1 \Rightarrow e^{-t}=\alpha_0-\alpha_1$$

$$f'(\lambda_1)=g'(\lambda_1)=\alpha_1 \Rightarrow te^{-t}=\alpha_1 \Rightarrow \alpha_0=e^{-t}+te^{-t}$$

then; $e^{At}=\alpha_0 I+\alpha_1 A=\begin{pmatrix}\alpha_0 & 0\\ 0 & \alpha_0\end{pmatrix}+\alpha_1\begin{pmatrix}0 & 1\\ -1 & -2\end{pmatrix}=\begin{pmatrix}\alpha_0 & \alpha_1\\ -\alpha_1 & \alpha_0-2\alpha_1\end{pmatrix}$

thus: $e^{At}=\begin{pmatrix}e^{-t}+te^{-t} & te^{-t}\\ -te^{-t} & e^{-t}-te^{-t}\end{pmatrix}$.

B) For a system with $A=\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}$, $B=\begin{bmatrix}0\\ 1\end{bmatrix}$, $C=[0 \quad 1]$, $D=0$, the matrix exponential e^{At} is found to be $\begin{bmatrix}3e^{-2t}-2e^{-3t} & e^{-2t}-e^{-3t}\\ -6e^{-2t}+6e^{-3t} & -2e^{-2t}+3e^{-3t}\end{bmatrix}$. It is required to find:

- System Transfer Function. (10 Marks)

$$T.F=C(sI-A)^{-1}B+D=[0 \quad 1]\begin{bmatrix}s & -1\\ 6 & s+5\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}=[0 \quad 1]\frac{\begin{bmatrix}s+5 & 1\\ -6 & s\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}}{s^2+5s+6}$$

thus; $T.F=\dfrac{s}{s^2+5s+6}$

- Initial and final values for unit impulse change in input. Discuss your results. (10 Marks)

for $U(s)=1$, $y(0)=\lim\limits_{s\to\infty} sy(s)=\lim\limits_{s\to\infty}\dfrac{s^2}{s^2+5s+6}=\lim\limits_{s\to\infty}\dfrac{1}{1+5/s+6/s^2}=1$.

$y(\infty)=\lim\limits_{s\to 0} sy(s)=\lim\limits_{s\to 0}\dfrac{s^2}{s^2+5s+6}=0$

The obtained results are compatible with initial and final unit impulse response values of a stable system.

- The output response $y(t)$ to a unit impulse change in input with zero initial conditions using the given state-space representation. (10 Marks)

$\underline{x}(t)=e^{At}\underline{x}(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$ and $y(t)=C\underline{x}(t)+Du$, for the given zero initial condition $\underline{x}(t)=\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$ and $y(t)=C\underline{x}(t)$ for given $D=0$

Now, for unit impulse change in input its effect is just at $t=0$, thus

$$\underline{x}(t)=\int_0^0 e^{A(t-\tau)}\begin{bmatrix}0\\ 1\end{bmatrix}[1]d\tau=\int_0^0\begin{matrix}e^{-2\tau}-e^{-3\tau}\\ -2e^{-2\tau}+3e^{-3\tau}\end{matrix}d\tau=\begin{bmatrix}-\frac{1}{2}e^{-2\tau}+\frac{1}{3}e^{-3\tau}\\ e^{-2\tau}-e^{-3\tau}\end{bmatrix}$$

$$y(t)=C\underline{x}(t)=[0 \quad 1]\underline{x}(t)=e^{-2t}-e^{-3t}$$

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Instructor: Dr. Mohammed Mahdi

Mid Exam. (30 Marks) Thursday: 5/12/2013

"Key Solutions"

Q1: (20 Marks): Evaluate the following using Laplace and z transforms formulas.

- Laplace Transform of $t^3 e^{-3t}$ (5 Ms)

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) = (-1)^n \frac{d^n}{ds^n} F(s)$$

since $\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$ then $\mathcal{L}\{t e^{-3t}\} = \frac{-d}{ds}\left(\frac{1}{(s+3)}\right) = \frac{1}{(s+3)^2}$

$\mathcal{L}\{t^2 e^{-3t}\} = \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right) = \frac{-2}{(s+3)^3}$ then $\mathcal{L}\{t^3 e^{-3t}\} = \frac{6}{(s+3)^4}$.

- Laplace inverse of $\frac{1}{s(s^2+1)}$ (5 Ms)

since $\mathcal{L}^{-1}\frac{1}{(s^2+1)} = \sin t$ then $\mathcal{L}^{-1}\frac{1}{s(s^2+1)} = \int_0^t \sin u \, du = -\cos u \Big|_0^t = 1-\cos t$.

- Prove that Laplace inverse of $F(s-a)$ is $e^{at} f(t)$. (5 Ms)

since $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ then $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

thus $F(s-a) = \int_0^\infty e^{-st}\{e^{at} f(t)\}\,dt = \mathcal{L}\{e^{at} f(t)\}$ then $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$.

- Prove that $z\{t\} = \frac{Tz}{(z-1)^2}$ (5 Ms)

$$Z\{t\}: X(z) = \sum_{k=0}^{k=\infty} z^{-k} kT = 0 + Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + 4Tz^{-4} + \cdots$$

$$z^{-1}X(z) = Tz^{-2} + 2Tz^{-3} + 3Tz^{-4} + \cdots$$

thus; $X(z) - z^{-1}X(z) = Tz^{-1} + Tz^{-2} + Tz^{-3} + Tz^{-4} + \cdots$

$(1-z^{-1})X(z) = \frac{Tz^{-1}}{1-z^{-1}}$ thus $X(z) = \frac{Tz^{-1}}{(1-z^{-1})^2}$ (Backward Representation)

and $X(z) = \frac{Tz}{(z-1)^2}$ (forward Representation)