Simplifying Logic Circuit

objectives:

1. Deriving of Logical expression from truth tables
2. Logical Expression simplification methods
   - 2.1. Algebraic manipulation
   - 2.2. Karnaugh map (K-map)

① Deriving of Logical expression from truth tables

Definitions:

- literal : non-complemented or complemented version of a variable (A or $\bar{A}$)
- product term : A series of literals related to one another through the AND operator (example: $A \cdot \bar{B} \cdot C$)
- Sum term : A series of literals related to one another through the OR operator. (EX : $A+B+\bar{C}$)
- SOP : (sum-of-products) form :

the logic-circuit simplification require the logic expression to be in SOP form, for example :

$AB + \bar{A}B\bar{C} + \bar{C}\bar{D}$: ⇒

- POS form (product-of-sum) :

this form sometimes used in logic circuit, example:

$(A+C) \cdot (C+\bar{D}) \cdot (\bar{B}+C)$.

The methods of circuit simplification and design that will be used are based on the SOP form.

- Canonical and standard form

• product terms that consist of the variables of function are called "canonical product terms" or "minterms".

The term $AB\bar{C}$ is a minterm in a three variable logic function, but will be a non-minterm in a four variable logic function.

• Sum terms which contain all the variables of a Boolean function are called "Canonical sum terms" or "Maxterms"

•Example : $A+\bar{B}+C$ → max term in a three variable logic function.

- For two variables A and B, there are four combination: $\bar{A}\bar{B}$, $\bar{A}B$, $A\bar{B}$, AB called minterm or standard product.

A	B	
0	0	$\rightarrow \bar{A}\bar{B}$
0	1	$\rightarrow \bar{A}B$
1	0	$\rightarrow A\bar{B}$
1	1	$\rightarrow AB$

- For n variables there are 2^n minterms.

Example: Minterms for 3 variables:

A	B	C	Minterm	Designation	Maxterm	Designation
0	0	0	$\bar{A}\bar{B}\bar{C}$	m_0	$A+B+C$	M_0
0	0	1	$\bar{A}\bar{B}C$	m_1	$A+B+\bar{C}$	M_1
0	1	0	$\bar{A}B\bar{C}$	m_2	$A+\bar{B}+C$	M_2
0	1	1	$\bar{A}BC$	m_3	$A+\bar{B}+\bar{C}$	M_3
1	0	0	$A\bar{B}\bar{C}$	m_4	$\bar{A}+B+C$	M_4
1	0	1	$A\bar{B}C$	m_5	$\bar{A}+B+\bar{C}$	M_5
1	1	0	$AB\bar{C}$	m_6	$\bar{A}+\bar{B}+C$	M_6
1	1	1	ABC	m_7	$\bar{A}+\bar{B}+\bar{C}$	M_7

Two important properties:

1. Any Boolean function can be expressed as a sum of minterms.
2. Any Boolean function can be expressed as a product of maxterms.

Example: majority function: (for 3 variables)

Output is one whenever majority of inputs is 1

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

- SOP form
- Four product terms, because there are 4 rows with a 1 output

$F = \bar{A}BC + A\bar{B}C$
$+ AB\bar{C}$
$+ ABC$

Final Expression:

$F = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

. pos form :

four sum terms, 4 rows with 0 output

$$F = (A+B+C)\cdot(A+B+\bar{C})\cdot(A+\bar{B}+C)\cdot(\bar{A}+B+C)$$

Derivation of Logical expression from truth tables " Summary"

- Sum-of-product (SOP) form.
- product-of-sums (POS) form.

- SOP form :
  . write an AND term for each input combination that produces a 1 output.
  . write the variable if its value is 1, Complement otherwise
  . OR the AND terms to get the final expression
- pos form :
  Dual of the SOP form.

2. Logical Expression Simplification Methods:

Two basic methods :

1. Algebraic manipulation : use Boolean Laws to simplify the expression : (difficult to use and don't know if you have the simplified form.
2. Karnaugh map (K- map) method
   - Graphical method,
   - ease to use
   - Can be used to simplify logical expressions with a few variables.

Algebraic manipulation : ⇒

Example 1: Design a logic circuit that has three inputs, A, B and C and whose output will be High only when a majority of the input are High . (complete Design Procedure)

Solution

Step 1: set up the truth table

" See previous section"

Step 2: write the AND term for each case where the output is a 1 : (see previous section)

Step 3: write the sum-of-products expression for the output

$$F = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

Step 4: simplify the output expression.

- find the common term(s): $\underline{ABC}$ - common term.
- use the common term (ABC) to factor with other terms.

Added extra

$$F = \bar{A}BC + ABC + A\bar{B}C + ABC + AB\bar{C} + ABC$$

$$\boxed{ABC + ABC + ABC = ABC}$$

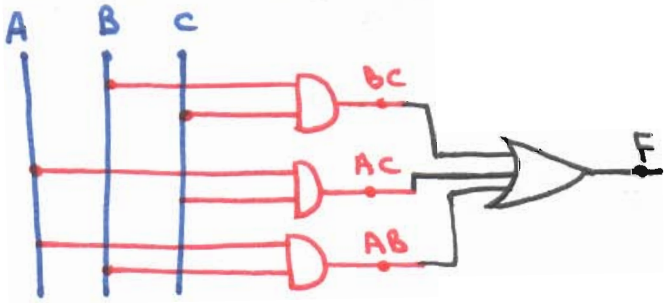
- Factoring the appropriate pairs of terms:

$$F = BC(\bar{A}+A) + AC(B+\bar{B}) + AB(\bar{C}+C)$$

$$\boxed{A + \bar{A} = 1}$$

$$F = BC + AC + AB$$

Step 5: Implement the circuit for the final expression



Example 2 : Simplify the boolean function

$$F = AB + \bar{A}C + BC$$

Solution: BC → common term

$$F = AB + \bar{A}C + BC \cdot 1$$

$$= AB + \bar{A}C + BC(A + \bar{A})$$

$$= AB + ABC + \bar{A}C + \bar{A}BC$$

$$= AB(1 + C) + \bar{A}C(1 + B)$$

$$= \underline{\underline{AB + \bar{A}C}}$$ the result.

Example 3: Simplify the following Boolean function to a minimum number of literals.

$$X = BC + A\bar{C} + AB + BCD$$

Solution:

$$X = BC(1+D) + A\bar{C} + AB$$

$$= BC + A\bar{C} + AB$$

Example 4: Simplify the expression

$$Z = A\underline{\bar{B}\bar{C}} + A\underline{\bar{B}C} + ABC$$

Solution: (The expression is in SOP form)

$$Z = A\bar{B}(\bar{C}+C) + ABC$$

$$= A\bar{B}(1) + ABC = A\bar{B} + ABC$$

factor A

$$Z = A(\bar{B} + BC) = \underline{\underline{A(\bar{B}+C)}}$$

Example 5: Simplify

$$Z = \bar{A}C(\overline{\bar{A}BD}) + \bar{A}B\bar{C}\bar{D} + A\bar{B}C$$

Solution:

First, use DeMorgan's theorem on the first term

$$Z = \bar{A}C(\bar{\bar{A}} + \bar{B} + \bar{D}) + \bar{A}B\bar{C}\bar{D} + A\bar{B}C$$

$$Z = \bar{A}C(A + \bar{B} + \bar{D}) + \bar{A}B\bar{C}\bar{D} + A\bar{B}C$$

multiply, we get:

$$Z = \underbrace{\bar{A}CA}_{=0} + \bar{A}C\bar{B} + \bar{A}C\bar{D} + \bar{A}B\bar{C}\bar{D} + A\bar{B}C$$

$\underline{\underline{\bar{A}.A=0}}$

common $(\bar{A}\bar{D})$ common $(\bar{B}C)$

$$Z = \bar{A}\bar{B}C + \bar{A}C\bar{D} + \bar{A}B\bar{C}\bar{D} + A\bar{B}C.$$

* check for the largest common factor between any two or more product terms.

$$Z = \bar{B}C(\underbrace{\bar{A} + A}_{=1}) + \bar{A}\bar{D}(\underbrace{C + B\bar{C}}_{=C+B})$$

we get

$Z = \bar{B}C + \bar{A}\bar{D}(B+C)$. (The result can be obtained with other choices.

Example 6 :

Find the complement of the following boolean function

$$Z = (B\bar{C} + \bar{A}D)(A\bar{B} + C\bar{D})$$

Solution: using DeMorgan's Law:

$$\bar{Z} = \overline{(B\bar{C} + \bar{A}D)(A\bar{B} + C\bar{D})} = \underbrace{\overline{B\bar{C} + \bar{A}D}}_{F_1} + \underbrace{\overline{A\bar{B} + C\bar{D}}}_{F_2}$$

$$= \overline{B\bar{C}} \cdot \overline{\bar{A}D} + \overline{A\bar{B}} \cdot \overline{C\bar{D}}$$

$$= (\bar{B} + \bar{\bar{C}}) \cdot (\bar{\bar{A}} + \bar{D}) + (\bar{A} + \bar{\bar{B}})(\bar{C} + \bar{\bar{D}})$$

$$= (\bar{B} + C)(A + \bar{D}) + (\bar{A} + B)(\bar{C} + D)$$

$$\boxed{\bar{\bar{A}} = A}$$

It's not in standard form.

Example 7 :

Express the following function in a sum of minterms and Product of maxterms.

$$Z = (AB + C)(B + AC)$$

- Sum of minterms :

multiply, we get :

the only minterm in equation

$$Z = ABB + AABC + BC + ACC = AB + \underline{ABC} + BC + AC$$

$$= ABC + AB(C + \bar{C}) + BC(A + \bar{A}) + AC(B + \bar{B})$$

$$\boxed{\begin{array}{l} C + \bar{C} = 1 \\ B + \bar{B} = 1 \\ A + \bar{A} = 1 \end{array}}$$

$$= \underline{ABC} + \underline{ABC} + AB\bar{C} + \underline{ABC} + \bar{A}BC + \underline{ABC} + A\bar{B}C$$

repeated terms

$$\boxed{A + A = A}$$

$$Z = \underset{\substack{\downarrow \\ m_7}}{ABC} + \underset{\substack{\downarrow \\ m_6}}{AB\bar{C}} + \underset{\substack{\downarrow \\ m_3}}{\bar{A}BC} + \underset{\substack{\downarrow \\ m_5}}{A\bar{B}C}$$

So, we can write :

$$Z(A, B, C) = \sum m(3, 5, 6, 7)$$

A	B	C	min term	Z
0	0	0	m_0 →	0
0	0	1	m_1 →	0
0	1	0	m_2 →	0
0	1	1	m_3 →	1
1	0	0	m_4 →	0
1	0	1	m_5 →	1
1	1	0	m_6 →	1
1	1	1	m_7 →	1

Sum of minterms

- sum of maxterms

using distributive law

$$\boxed{(A + BC) \Rightarrow (A + B)(A + C)}$$

in the $Z = (AB + C)(B + AC)$

we get :

$$= (A + C)\underline{(B + C)}(B + A)\underline{\cancel{(B + C)}}$$

$$= (A + B)(B + C)(C + A)$$

we need $\underline{\underline{C}}$

$$Z = (A+B)(B+C)(C+A)$$

we need $\underline{\underline{A}}$ — we need $\underline{\underline{B}}$

We can write

$$(A+B) = (A+B+0) = (A+B+C\bar{C}) = \boxed{(A+B+C)(A+B+\bar{C})}$$

$$(B+C) = (B+C+0) = \boxed{(B+C+A)(B+C+\bar{A})}$$

$$(C+A) = \boxed{(C+A+B)(C+A+\bar{B})}$$

substitue all of these terms in Z, we get:

$$Z = (A+B)(B+C)(C+A) = (A+B+C)(A+B+\bar{C})\cancel{(B+C+A)}(B+C+\bar{A})\cancel{(C+A+B)}(C+A+\bar{B})$$

$$Z = (A+B+C)(A+B+\bar{C})(B+C+\bar{A})(C+A+\bar{B})$$

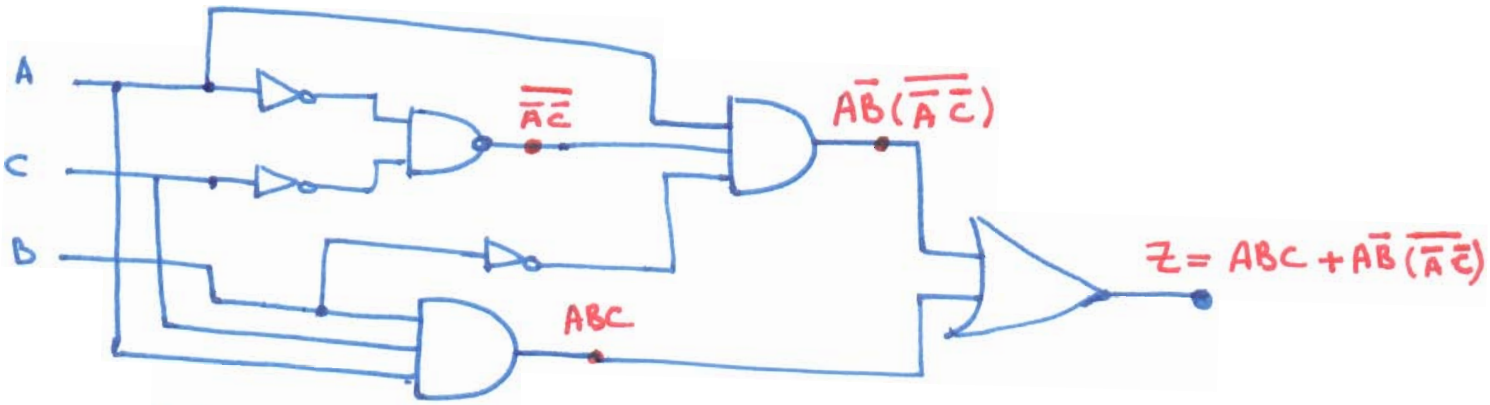
$$Z = \underbrace{(A+B+C)}_{M_0}\underbrace{(A+B+\bar{C})}_{M_1}\underbrace{(\bar{A}+B+C)}_{M_4}\underbrace{(A+\bar{B}+C)}_{M_2}$$

$$Z = M_0 \cdot M_1 \cdot M_4 \cdot M_2$$

A	B	C	max term	Z
0	0	0	M_0	0
0	0	1	M_1	0
0	1	0	M_2	0
0	1	1	M_3	1
1	0	0	M_4	0
1	0	1	M_5	1
1	1	0	M_6	1
1	1	1	M_7	1

Sum of maxterms

Example 8 : simplify the Logic circuit shown in figure.



Solution :

$Z = ABC + A\bar{B}\cdot\overline{(\bar{A}\bar{C})}$ ⇒ using DeMorgan's Law and multiply out all terms :

$Z = ABC + A\bar{B}(\bar{\bar{A}} + \bar{\bar{C}})$ → "DeMorgan's Law"

$Z = ABC + A\bar{B}(A + C)$ → "Cancel double inversion"

$Z = ABC + A\bar{B}A + A\bar{B}C$ → "multiply out"

$Z = ABC + A\bar{B} + A\bar{B}C$ → "$A \cdot A = A$"

↓

SOP form

$Z = AC(B+\bar{B}) + A\bar{B}$ → "Common terms and factoring"

$Z = AC\cdot 1 + A\bar{B} = AC + A\bar{B} = A(C+\bar{B})$