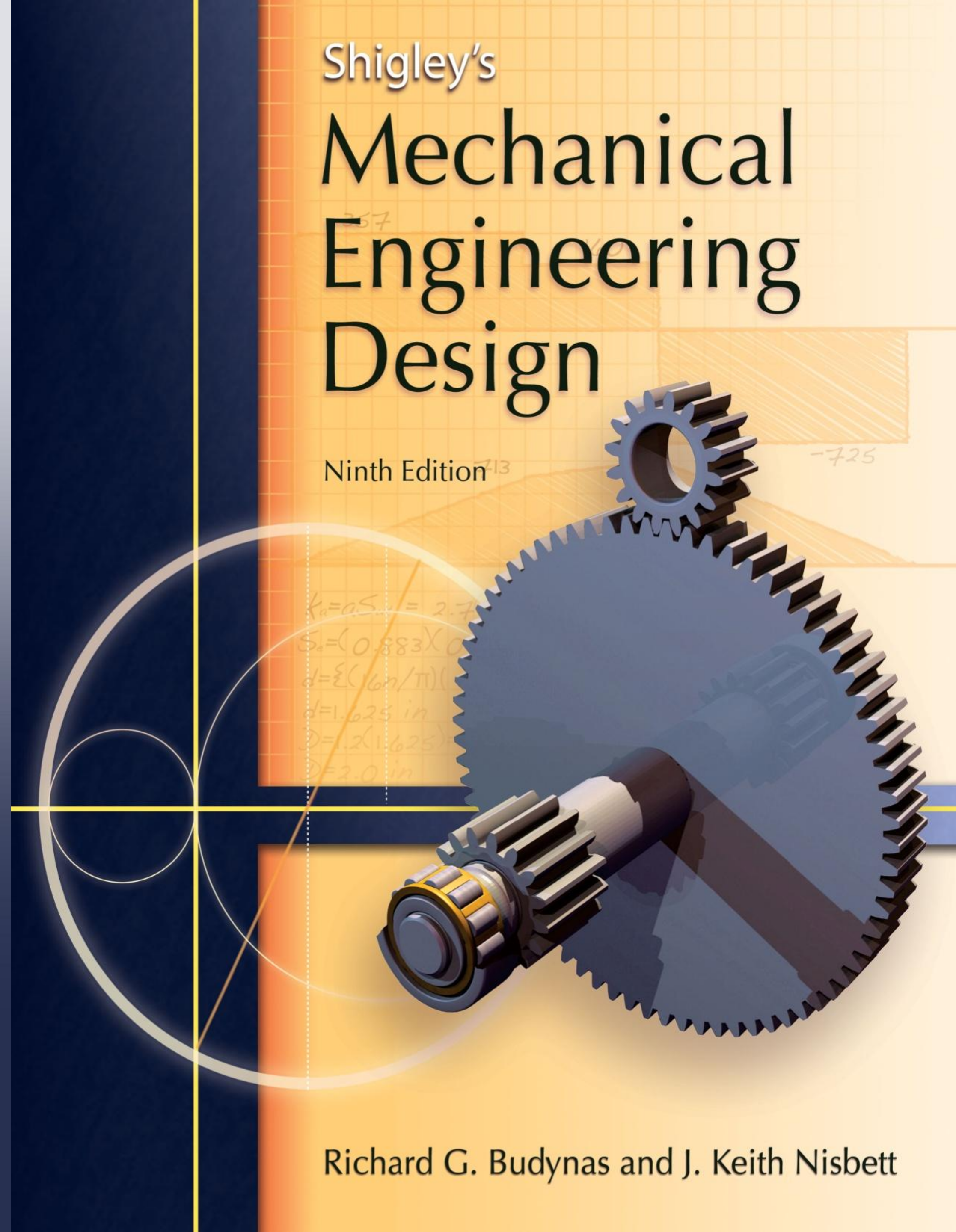
Lecture Slides

Chapter 4

Deflection and Stiffness



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Force vs Deflection

- *Elasticity* – property of a material that enables it to regain its original configuration after deformation
- *Spring* – a mechanical element that exerts a force when deformed

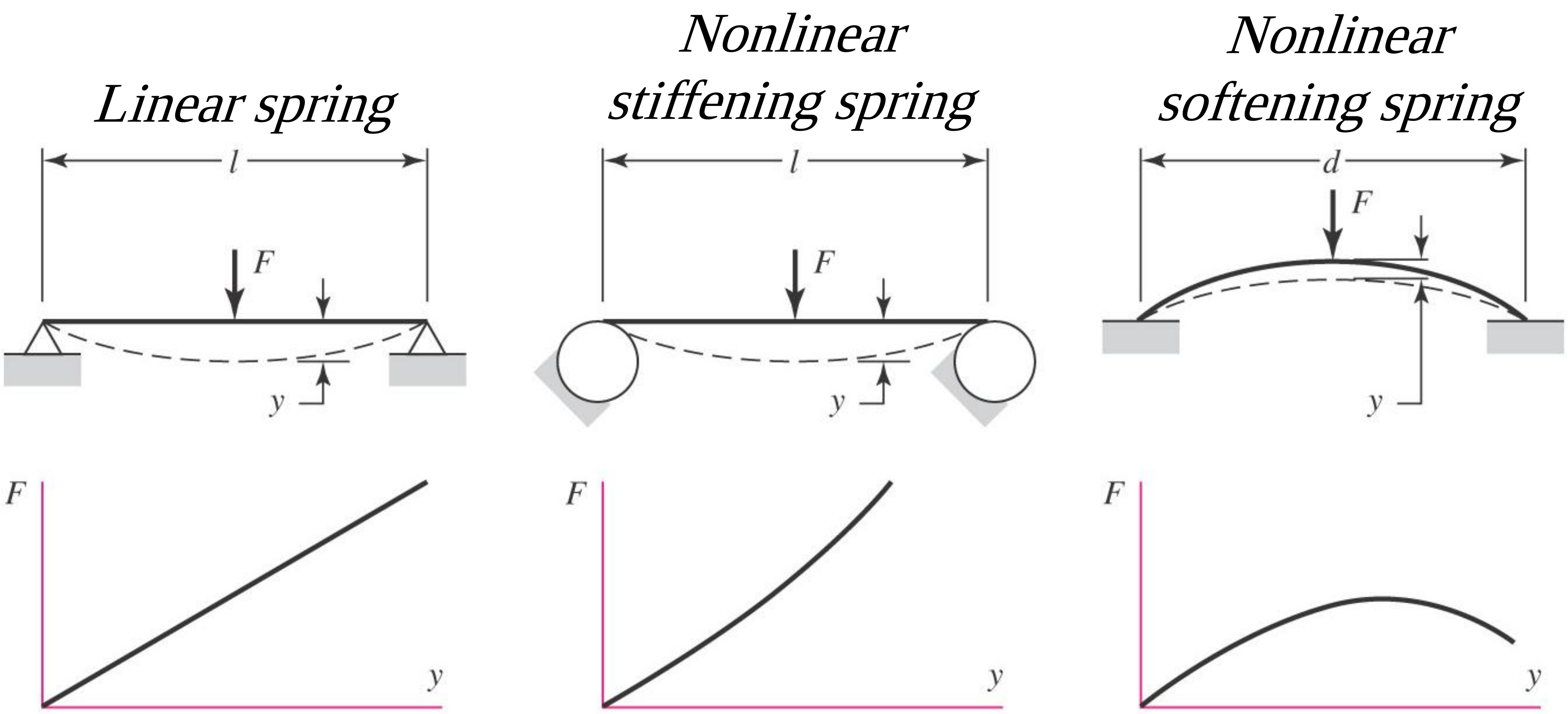


Fig. 4–1

Spring Rate

- Relation between force and deflection, $F = F(y)$
- *Spring rate*

$$k(y) = \lim_{\Delta y \to 0} \frac{\Delta F}{\Delta y} = \frac{dF}{dy} \qquad (4\text{–}1)$$

- For linear springs, k is constant, called *spring constant*

$$k = \frac{F}{y} \qquad (4\text{–}2)$$

Axially-Loaded Stiffness

- Total extension or contraction of a uniform bar in tension or compression

$$\delta = \frac{Fl}{AE} \tag{4-3}$$

- Spring constant, with $k = F/\delta$

$$k = \frac{AE}{l} \tag{4-4}$$

Torsionally-Loaded Stiffness

- Angular deflection (in radians) of a uniform solid or hollow round bar subjected to a twisting moment T

$$\theta = \frac{Tl}{GJ} \qquad (4\text{–}5)$$

- Converting to degrees, and including $J = \pi d^4/32$ for round solid

$$\theta = \frac{583.6Tl}{Gd^4} \qquad (4\text{–}6)$$

- Torsional spring constant for round bar

$$k = \frac{T}{\theta} = \frac{GJ}{l} \qquad (4\text{–}7)$$

Deflection Due to Bending

- Curvature of beam subjected to bending moment M

$$\frac{1}{\rho} = \frac{M}{EI} \qquad (4\text{–}8)$$

- From mathematics, curvature of plane curve

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \qquad (4\text{–}9)$$

- Slope of beam at any point x along the length

$$\theta = \frac{dy}{dx}$$

- If the slope is very small, the denominator of Eq. (4-9) approaches unity.
- Combining Eqs. (4-8) and (4-9), for beams with small slopes,

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

Deflection Due to Bending

- Recall Eqs. (3-3) and (3-4)

$$V = \frac{dM}{dx} \qquad (3\text{–}3)$$

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = q \qquad (3\text{–}4)$$

- Successively differentiating

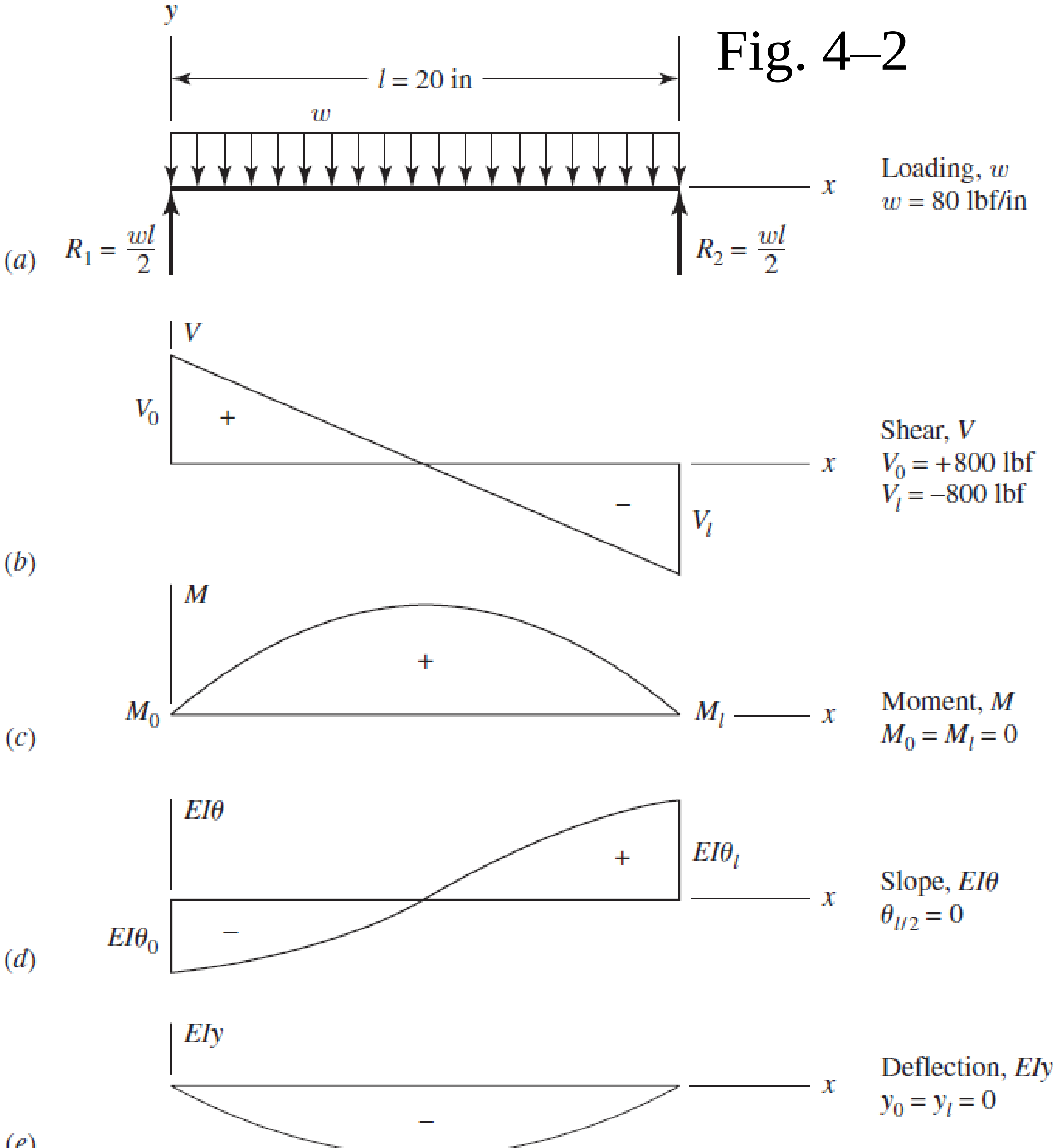
$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

$$\frac{q}{EI} = \frac{d^4y}{dx^4}$$

Deflection Due to Bending

Fig. 4–2



$$\frac{q}{EI} = \frac{d^4 y}{dx^4} \qquad (4\text{-}10)$$

$$\frac{V}{EI} = \frac{d^3 y}{dx^3} \qquad (4\text{-}11)$$

$$\frac{M}{EI} = \frac{d^2 y}{dx^2} \qquad (4\text{-}12)$$

$$\theta = \frac{dy}{dx} \qquad (4\text{-}13)$$

$$y = f(x) \qquad (4\text{-}14)$$

Example 4-1

For the beam in Fig. 4–2, the bending moment equation, for $0 \le x \le l$, is

$$M = \frac{wl}{2}x - \frac{w}{2}x^2$$

Using Eq. (4–12), determine the equations for the slope and deflection of the beam, the slopes at the ends, and the maximum deflection.

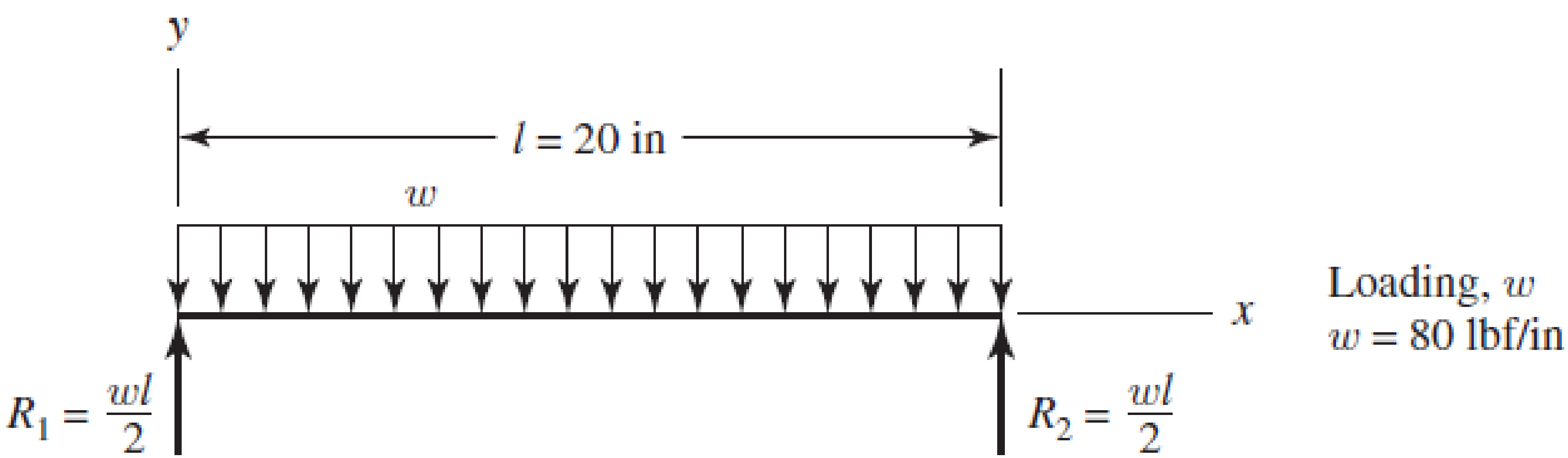


Fig. 4–2

Example 4-1

Integrating Eq. (4–12) as an indefinite integral we have

$$EI\frac{dy}{dx} = \int M\,dx = \frac{wl}{4}x^2 - \frac{w}{6}x^3 + C_1 \qquad (1)$$

where C_1 is a constant of integration that is evaluated from geometric boundary conditions. We could impose that the slope is zero at the midspan of the beam, since the beam and loading are symmetric relative to the midspan. However, we will use the given boundary conditions of the problem and verify that the slope is zero at the midspan. Integrating Eq. (1) gives

$$EIy = \iint M\,dx = \frac{wl}{12}x^3 - \frac{w}{24}x^4 + C_1x + C_2 \qquad (2)$$

The boundary conditions for the simply supported beam are $y = 0$ at $x = 0$ and l. Applying the first condition, $y = 0$ at $x = 0$, to Eq. (2) results in $C_2 = 0$. Applying the second condition to Eq. (2) with $C_2 = 0$,

$$EIy(l) = \frac{wl}{12}l^3 - \frac{w}{24}l^4 + C_1 l = 0$$

Example 4-1

Solving for C_1 yields $C_1 = -wl^3/24$. Substituting the constants back into Eqs. (1) and (2) and solving for the deflection and slope results in

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3) \tag{3}$$

$$\theta = \frac{dy}{dx} = \frac{w}{24EI}(6lx^2 - 4x^3 - l^3) \tag{4}$$

Comparing Eq. (3) with that given in Table A–9, beam 7, we see complete agreement. For the slope at the left end, substituting $x = 0$ into Eq. (4) yields

$$\theta|_{x=0} = -\frac{wl^3}{24EI}$$

and at $x = l$,

$$\theta|_{x=l} = \frac{wl^3}{24EI}$$

At the midspan, substituting $x = l/2$ gives $dy/dx = 0$, as earlier suspected.

Example 4-1

The maximum deflection occurs where $dy/dx = 0$. Substituting $x = l/2$ into Eq. (3) yields

$$y_{max} = -\frac{5wl^4}{384EI}$$

which again agrees with Table A–9–7.

Beam Deflection Methods

- Some of the more common methods for solving the integration problem for beam deflection
  - Superposition
  - Moment-area method
  - Singularity functions
  - Numerical integration
- Other methods that use alternate approaches
  - Castigliano energy method
  - Finite element software

Beam Deflection by Superposition

- *Superposition* determines the effects of each load separately, then adds the results.
- Separate parts are solved using any method for simple load cases.
- Many load cases and boundary conditions are solved and available in Table A-9, or in references such as *Roark's Formulas for Stress and Strain.*
- Conditions
  - Each effect is linearly related to the load that produces it.
  - A load does not create a condition that affects the result of another load.
  - The deformations resulting from any specific load are not large enough to appreciably alter the geometric relations of the parts of the structural system.

Example 4-2

Consider the uniformly loaded beam with a concentrated force as shown in Fig. 4–3. Using superposition, determine the reactions and the deflection as a function of x.

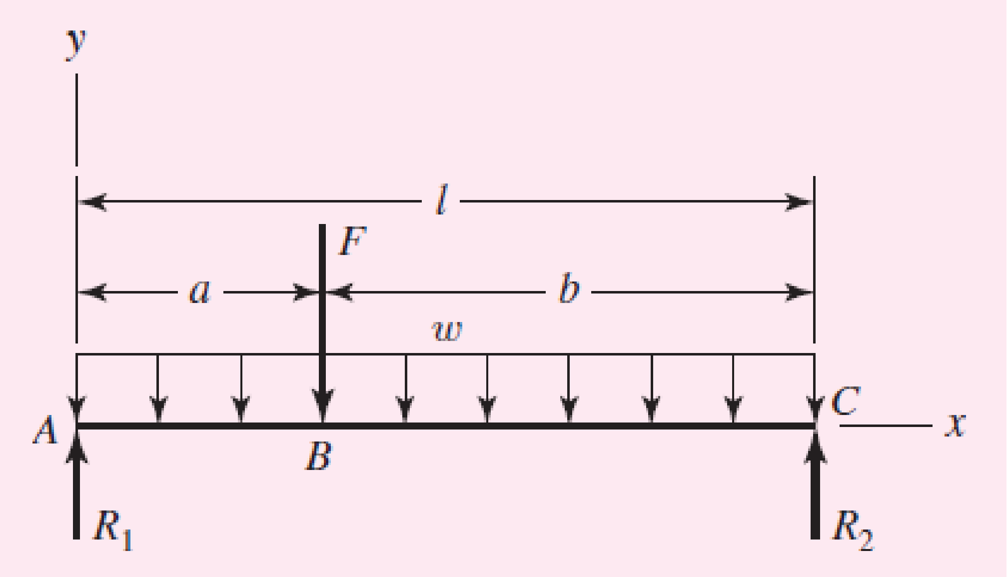
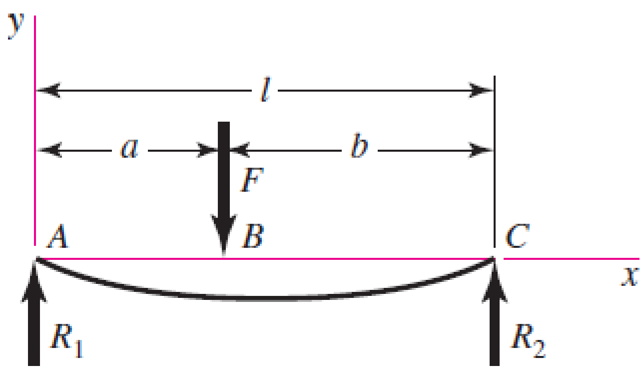


Fig. 4–3

Example 4-2

Considering each load state separately, we can superpose beams 6 and 7 of Table A–9.

6 Simple supports—intermediate load



$$R_1 = \frac{Fb}{l} \qquad R_2 = \frac{Fa}{l}$$

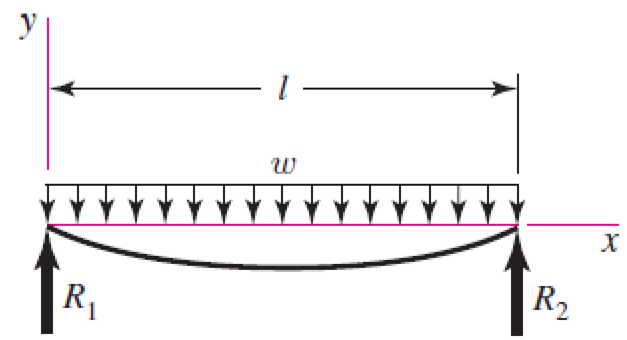
$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fbx}{l} \qquad M_{BC} = \frac{Fa}{l}(l - x)$$

$$y_{AB} = \frac{Fbx}{6EIl}(x^2 + b^2 - l^2)$$

$$y_{BC} = \frac{Fa(l - x)}{6EIl}(x^2 + a^2 - 2lx)$$

7 Simple supports—uniform load



$$R_1 = R_2 = \frac{wl}{2} \qquad V = \frac{wl}{2} - wx$$

$$M = \frac{wx}{2}(l - x)$$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{max} = -\frac{5wl^4}{384EI}$$

Example 4-2

For the reactions we find

$$R_1 = \frac{Fb}{l} + \frac{wl}{2}$$

$$R_2 = \frac{Fa}{l} + \frac{wl}{2}$$

The loading of beam 6 is discontinuous and separate deflection equations are given for regions AB and BC. Beam 7 loading is not discontinuous so there is only one equation. Superposition yields

$$y_{AB} = \frac{Fbx}{6EIl}(x^2 + b^2 - l^2) + \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{BC} = \frac{Fa(l-x)}{6EIl}(x^2 + a^2 - 2lx) + \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

Example 4-3

Consider the beam in Fig. 4–4*a* and determine the deflection equations using superposition.

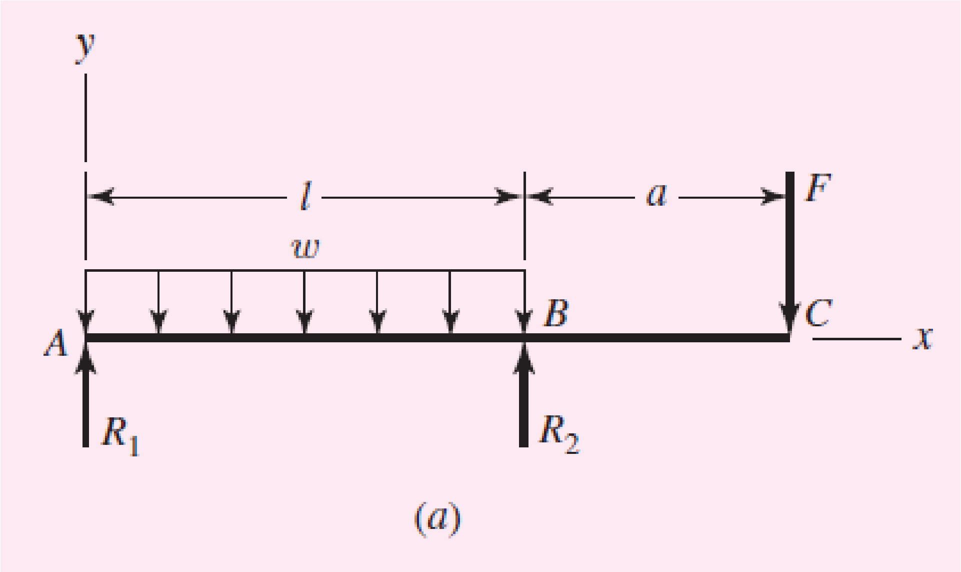


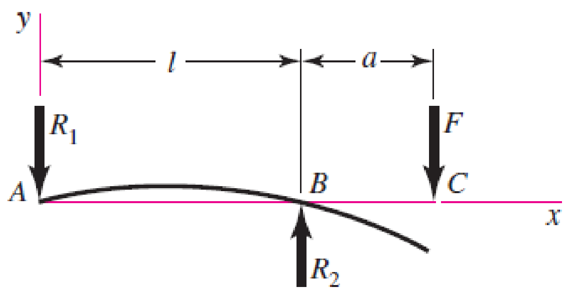
Fig. 4–4

Example 4-3

For region AB we can superpose beams 7 and 10 of Table A–9 to obtain

$$y_{AB} = \frac{wx}{24EI}(2lx^2 - x^3 - l^3) + \frac{Fax}{6EIl}(l^2 - x^2)$$

10 Simple supports—overhanging load



$$R_1 = \frac{Fa}{l} \qquad R_2 = \frac{F}{l}(l + a)$$

$$V_{AB} = -\frac{Fa}{l} \qquad V_{BC} = F$$

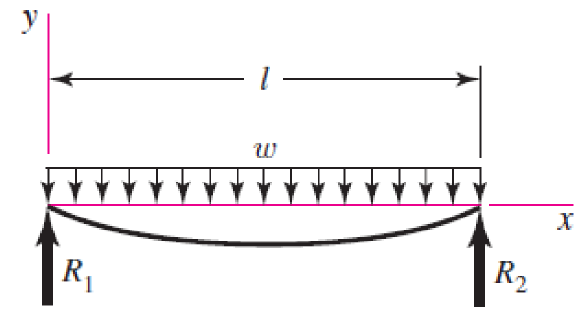
$$M_{AB} = -\frac{Fax}{l} \qquad M_{BC} = F(x - l - a)$$

$$y_{AB} = \frac{Fax}{6EIl}(l^2 - x^2)$$

$$y_{BC} = \frac{F(x-l)}{6EI}[(x-l)^2 - a(3x - l)]$$

$$y_C = -\frac{Fa^2}{3EI}(l + a)$$

7 Simple supports—uniform load



$$R_1 = R_2 = \frac{wl}{2} \qquad V = \frac{wl}{2} - wx$$

$$M = \frac{wx}{2}(l - x)$$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{max} = -\frac{5wl^4}{384EI}$$

Example 4-3

For region BC, how do we represent the uniform load? Considering the uniform load *only*, the beam deflects as shown in Fig. 4–4*b*. Region BC is straight since there is no bending moment due to w. The slope of the beam at B is θ_B and is obtained by taking the derivative of y given in the table with respect to x and setting $x = l$. Thus,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{wx}{24EI}(2lx^2 - x^3 - l^3)\right] = \frac{w}{24EI}(6lx^2 - 4x^3 - l^3)$$

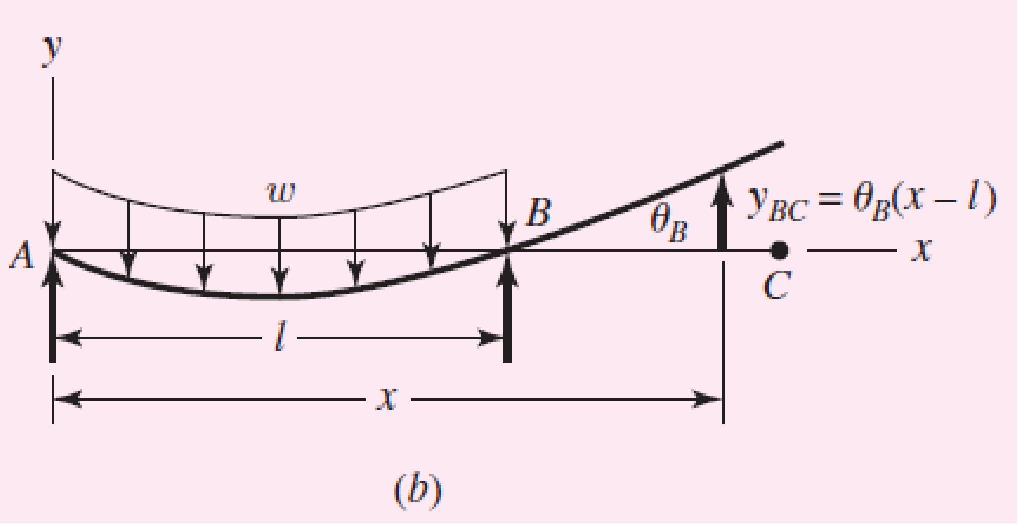


Fig. 4–4

Example 4-3

Substituting $x = l$ gives

$$\theta_B = \frac{w}{24EI}(6ll^2 - 4l^3 - l^3) = \frac{wl^3}{24EI}$$

The deflection in region BC due to w is $\theta_B(x - l)$, and adding this to the deflection due to F, in BC, yields

$$y_{BC} = \frac{wl^3}{24EI}(x - l) + \frac{F(x - l)}{6EI}[(x - l)^2 - a(3x - l)]$$

Example 4-4

Figure 4–5a shows a cantilever beam with an end load. Normally we model this problem by considering the left support as rigid. After testing the rigidity of the wall it was found that the translational stiffness of the wall was k_t force per unit vertical deflection, and the rotational stiffness was k_r moment per unit angular (radian) deflection (see Fig. 4–5b). Determine the deflection equation for the beam under the load F.

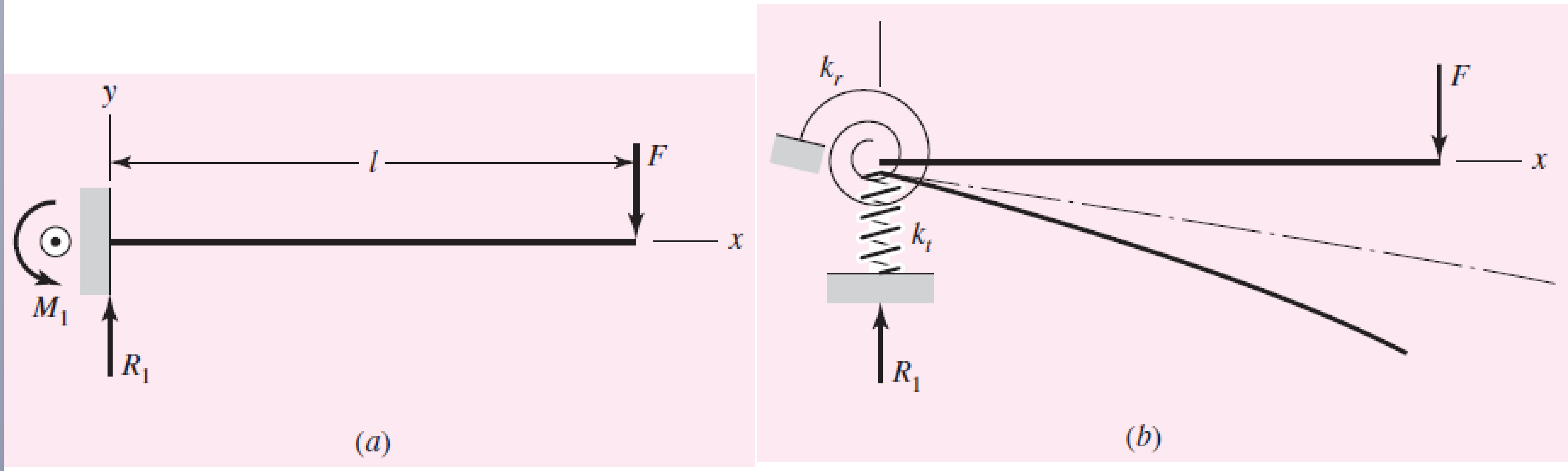


Fig. 4–5

Example 4-4

Here we will superpose the *modes* of deflection. They are: (1) translation due to the compression of spring k_t, (2) rotation of the spring k_r, and (3) the elastic deformation of beam 1 given in Table A–9. The force in spring k_t is $R_1 = F$, giving a deflection from Eq. (4–2) of

$$y_1 = -\frac{F}{k_t} \tag{1}$$

The moment in spring k_r is $M_1 = Fl$. This gives a clockwise rotation of $\theta = Fl/k_r$. Considering this mode of deflection only, the beam rotates rigidly clockwise, leading to a deflection equation of

$$y_2 = -\frac{Fl}{k_r}x \tag{2}$$

Example 4-4

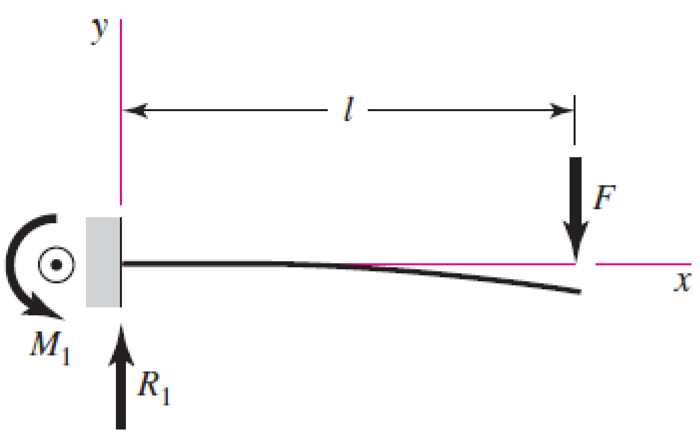
Finally, the elastic deformation of beam 1 from Table A–9 is

$$y_3 = \frac{Fx^2}{6EI}(x - 3l) \tag{3}$$

Adding the deflections from each mode yields

$$y = \frac{Fx^2}{6EI}(x - 3l) - \frac{F}{k_t} - \frac{Fl}{k_r}x$$

1 Cantilever—end load



$$R_1 = V = F \qquad M_1 = Fl$$

$$M = F(x - l)$$

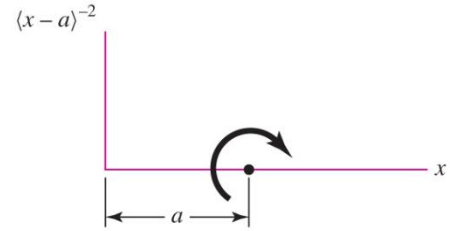
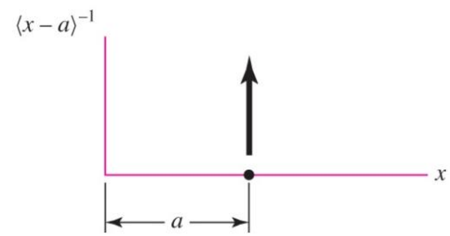
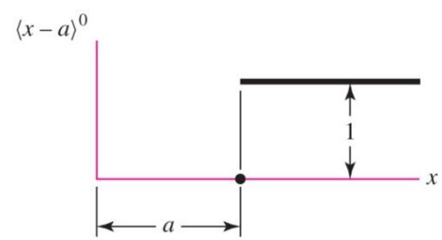
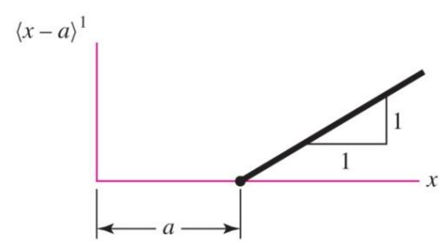
$$y = \frac{Fx^2}{6EI}(x - 3l)$$

$$y_{max} = -\frac{Fl^3}{3EI}$$

Beam Deflection by Singularity Functions

- A notation useful for integrating across discontinuities
- Angle brackets indicate special function to determine whether forces and moments are active

Table 3–1

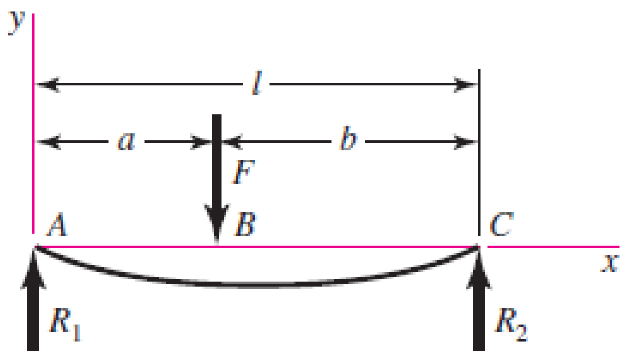
Function	Graph of $f_n(x)$	Meaning
Concentrated moment (unit doublet)		$\langle x-a\rangle^{-2}=0 \quad x\neq a$ $\langle x-a\rangle^{-2}=\pm\infty \quad x=a$ $\int \langle x-a\rangle^{-2}\,dx=\langle x-a\rangle^{-1}$
Concentrated force (unit impulse)		$\langle x-a\rangle^{-1}=0 \quad x\neq a$ $\langle x-a\rangle^{-1}=+\infty \quad x=a$ $\int \langle x-a\rangle^{-1}\,dx=\langle x-a\rangle^{0}$
Unit step		$\langle x-a\rangle^{0}=\begin{cases}0 & x<a\\ 1 & x\geq a\end{cases}$ $\int \langle x-a\rangle^{0}\,dx=\langle x-a\rangle^{1}$
Ramp		$\langle x-a\rangle^{1}=\begin{cases}0 & x<a\\ x-a & x\geq a\end{cases}$ $\int \langle x-a\rangle^{1}\,dx=\frac{\langle x-a\rangle^{2}}{2}$

†W. H. Macaulay, "Note on the deflection of beams," *Messenger of Mathematics*, vol. 48, pp. 129–130, 1919.

Example 4-5

Consider beam 6 of Table A–9, which is a simply supported beam having a concentrated load F not in the center. Develop the deflection equations using singularity functions.

6 Simple supports—intermediate load



$$R_1 = \frac{Fb}{l} \qquad R_2 = \frac{Fa}{l}$$

$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fbx}{l} \qquad M_{BC} = \frac{Fa}{l}(l - x)$$

$$y_{AB} = \frac{Fbx}{6EIl}(x^2 + b^2 - l^2)$$

$$y_{BC} = \frac{Fa(l - x)}{6EIl}(x^2 + a^2 - 2lx)$$

Example 4-5

First, write the load intensity equation from the free-body diagram,

$$q = R_1\langle x\rangle^{-1} - F\langle x-a\rangle^{-1} + R_2\langle x-l\rangle^{-1} \tag{1}$$

Integrating Eq. (1) twice results in

$$V = R_1\langle x\rangle^{0} - F\langle x-a\rangle^{0} + R_2\langle x-l\rangle^{0} \tag{2}$$

$$M = R_1\langle x\rangle^{1} - F\langle x-a\rangle^{1} + R_2\langle x-l\rangle^{1} \tag{3}$$

Recall that as long as the q equation is complete, integration constants are unnecessary for V and M; therefore, they are not included up to this point. From statics, setting $V = M = 0$ for x slightly greater than l yields $R_1 = Fb/l$ and $R_2 = Fa/l$. Thus Eq. (3) becomes

$$M = \frac{Fb}{l}\langle x\rangle^{1} - F\langle x-a\rangle^{1} + \frac{Fa}{l}\langle x-l\rangle^{1}$$

Example 4-5

Integrating Eqs. (4–12) and (4–13) as indefinite integrals gives

$$EI\frac{dy}{dx} = \frac{Fb}{2l}\langle x\rangle^2 - \frac{F}{2}\langle x-a\rangle^2 + \frac{Fa}{2l}\langle x-l\rangle^2 + C_1$$

$$EIy = \frac{Fb}{6l}\langle x\rangle^3 - \frac{F}{6}\langle x-a\rangle^3 + \frac{Fa}{6l}\langle x-l\rangle^3 + C_1x + C_2$$

Note that the first singularity term in both equations always exists, so $\langle x\rangle^2 = x^2$ and $\langle x\rangle^3 = x^3$. Also, the last singularity term in both equations does not exist until $x = l$, where it is zero, and since there is no beam for $x > l$ we can drop the last term. Thus

$$EI\frac{dy}{dx} = \frac{Fb}{2l}x^2 - \frac{F}{2}\langle x-a\rangle^2 + C_1 \tag{4}$$

$$EIy = \frac{Fb}{6l}x^3 - \frac{F}{6}\langle x-a\rangle^3 + C_1x + C_2 \tag{5}$$

Example 4-5

The constants of integration C_1 and C_2 are evaluated by using the two boundary conditions $y = 0$ at $x = 0$ and $y = 0$ at $x = l$. The first condition, substituted into Eq. (5), gives $C_2 = 0$ (recall that $\langle 0 - a \rangle^3 = 0$). The second condition, substituted into Eq. (5), yields

$$0 = \frac{Fb}{6l} l^3 - \frac{F}{6}(l-a)^3 + C_1 l = \frac{Fbl^2}{6} - \frac{Fb^3}{6} + C_1 l$$

Solving for C_1 gives

$$C_1 = -\frac{Fb}{6l}(l^2 - b^2)$$

Finally, substituting C_1 and C_2 in Eq. (5) and simplifying produces

$$y = \frac{F}{6EIl}\left[bx(x^2 + b^2 - l^2) - l\langle x - a \rangle^3\right] \tag{6}$$

Comparing Eq. (6) with the two deflection equations for beam 6 in Table A–9, we note that the use of singularity functions enables us to express the deflection equation with a single equation.

Example 4-6

Determine the deflection equation for the simply supported beam with the load distribution shown in Fig. 4–6.

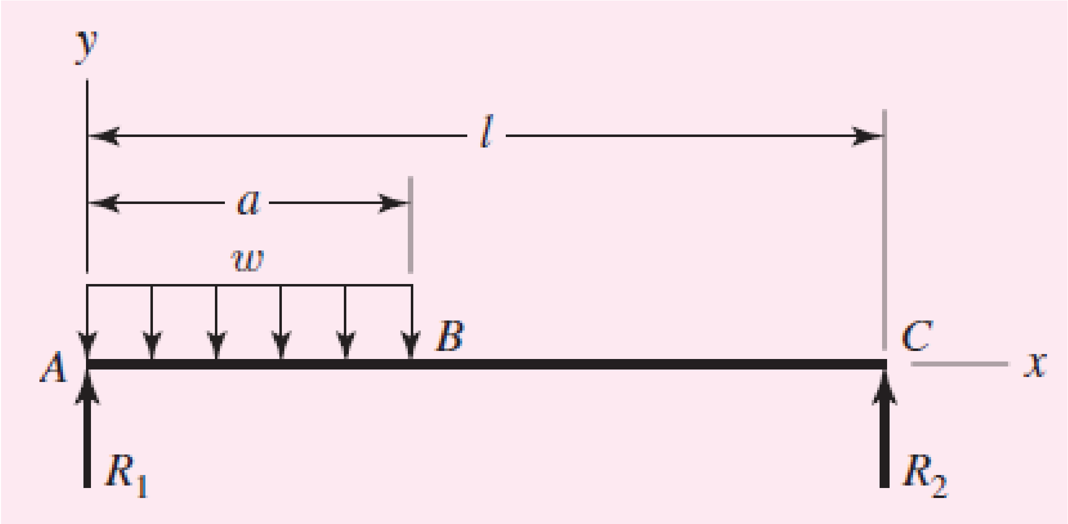


Fig. 4–6

Example 4-6

This is a good beam to add to our table for later use with superposition. The load intensity equation for the beam is

$$q = R_1\langle x\rangle^{-1} - w\langle x\rangle^0 + w\langle x-a\rangle^0 + R_2\langle x-l\rangle^{-1} \quad (1)$$

where the $w\langle x-a\rangle^0$ is necessary to "turn off" the uniform load at $x = a$.

From statics, the reactions are

$$R_1 = \frac{wa}{2l}(2l-a) \qquad R_2 = \frac{wa^2}{2l} \quad (2)$$

For simplicity, we will retain the form of Eq. (1) for integration and substitute the values of the reactions in later.

Two integrations of Eq. (1) reveal

$$V = R_1\langle x\rangle^0 - w\langle x\rangle^1 + w\langle x-a\rangle^1 + R_2\langle x-l\rangle^0 \quad (3)$$

$$M = R_1\langle x\rangle^1 - \frac{w}{2}\langle x\rangle^2 + \frac{w}{2}\langle x-a\rangle^2 + R_2\langle x-l\rangle^1 \quad (4)$$

Example 4-6

As in the previous example, singularity functions of order zero or greater starting at $x = 0$ can be replaced by normal polynomial functions. Also, once the reactions are determined, singularity functions starting at the extreme right end of the beam can be omitted. Thus, Eq. (4) can be rewritten as

$$M = R_1 x - \frac{w}{2}x^2 + \frac{w}{2}\langle x - a\rangle^2 \tag{5}$$

Integrating two more times for slope and deflection gives

$$EI\frac{dy}{dx} = \frac{R_1}{2}x^2 - \frac{w}{6}x^3 + \frac{w}{6}\langle x - a\rangle^3 + C_1 \tag{6}$$

$$EIy = \frac{R_1}{6}x^3 - \frac{w}{24}x^4 + \frac{w}{24}\langle x - a\rangle^4 + C_1 x + C_2 \tag{7}$$

Example 4-6

The boundary conditions are $y = 0$ at $x = 0$ and $y = 0$ at $x = l$. Substituting the first condition in Eq. (7) shows $C_2 = 0$. For the second condition

$$0 = \frac{R_1}{6} l^3 - \frac{w}{24} l^4 + \frac{w}{24}(l - a)^4 + C_1 l$$

Solving for C_1 and substituting into Eq. (7) yields

$$EIy = \frac{R_1}{6} x(x^2 - l^2) - \frac{w}{24} x(x^3 - l^3) - \frac{w}{24l} x(l - a)^4 + \frac{w}{24}\langle x - a\rangle^4$$

Finally, substitution of R_1 from Eq. (2) and simplifying results gives

$$y = \frac{w}{24EIl}[2ax(2l - a)(x^2 - l^2) - xl(x^3 - l^3) - x(l - a)^4 + l\langle x - a\rangle^4]$$

Example 4-7

The steel step shaft shown in Fig. 4–7*a* is mounted in bearings at *A* and *F*. A pulley is centered at *C* where a total radial force of 600 lbf is applied. Using singularity functions evaluate the shaft displacements at $\frac{1}{2}$-in increments. Assume the shaft is simply supported.

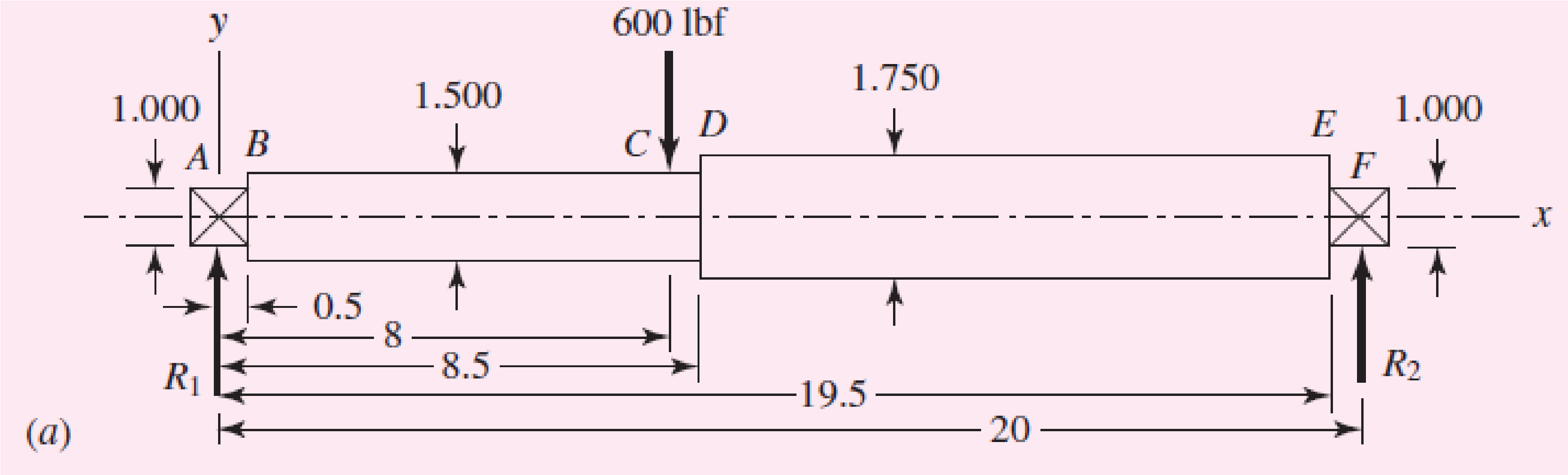


Fig. 4–7

Example 4-7

The reactions are found to be $R_1 = 360$ lbf and $R_2 = 240$ lbf. Ignoring R_2, using singularity functions, the moment equation is

$$M = 360x - 600\langle x - 8\rangle^1 \quad (1)$$

This is plotted in Fig. 4–7*b*.

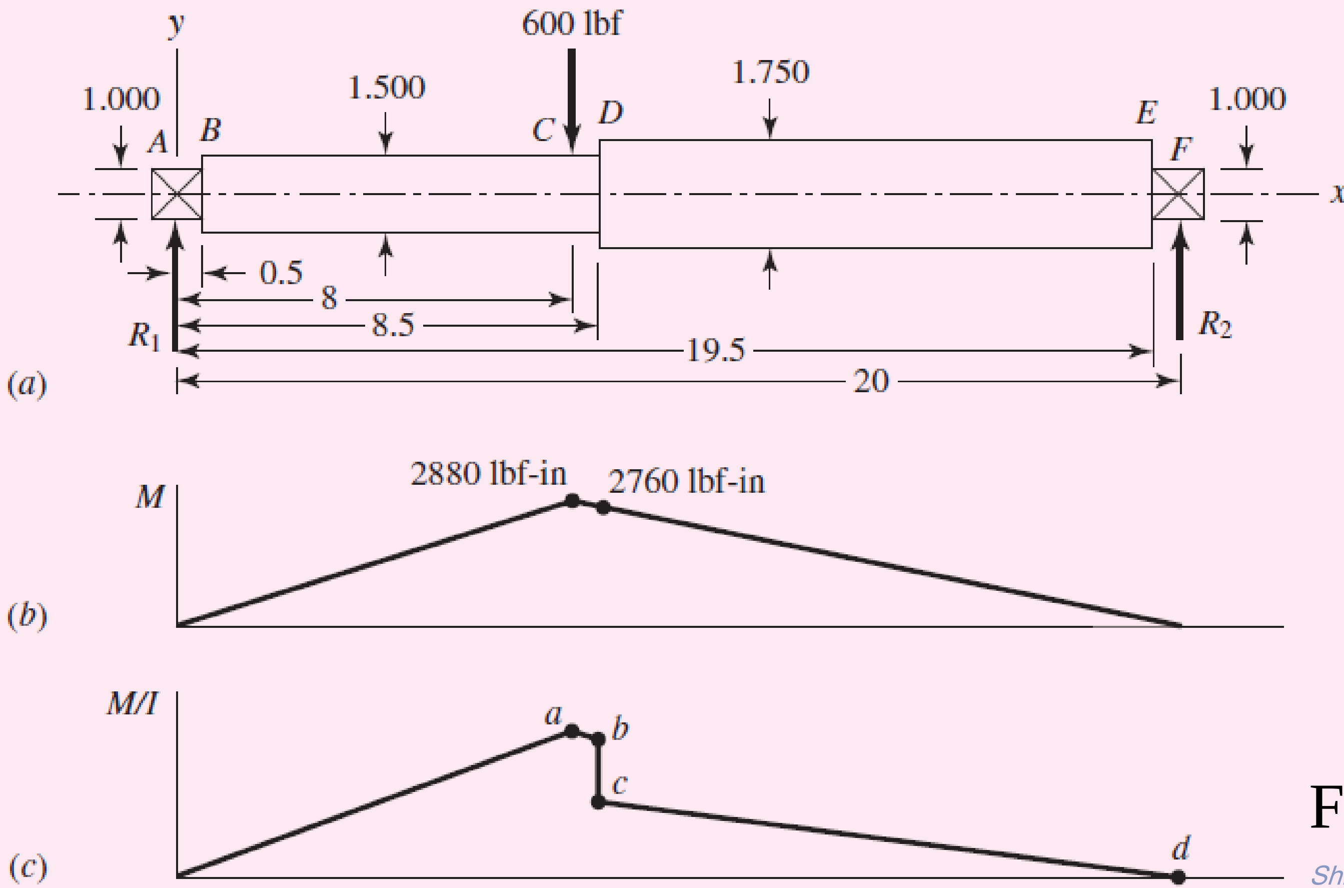


Fig. 4–7

Example 4-7

For simplification, we will consider only the step at D. That is, we will assume section AB has the same diameter as BC and section EF has the same diameter as DE. Since these sections are short and at the supports, the size reduction will not add much to the deformation. We will examine this simplification later. The second area moments for BC and DE are

$$I_{BC} = \frac{\pi}{64} 1.5^4 = 0.2485 \text{ in}^4 \qquad I_{DE} = \frac{\pi}{64} 1.75^4 = 0.4604 \text{ in}^4$$

A plot of M/I is shown in Fig. 4–7c. The values at points b and c, and the step change are

$$\left(\frac{M}{I}\right)_b = \frac{2760}{0.2485} = 11\,106.6 \text{ lbf/in}^3 \qquad \left(\frac{M}{I}\right)_c = \frac{2760}{0.4604} = 5\,994.8 \text{ lbf/in}^3$$

$$\Delta\left(\frac{M}{I}\right) = 5\,994.8 - 11\,106.6 = -5\,111.8 \text{ lbf/in}^3$$

Example 4-7

The slopes for ab and cd, and the change are

$$m_{ab} = \frac{2760 - 2880}{0.2485(0.5)} = -965.8 \text{ lbf/in}^4 \qquad m_{cd} = \frac{-5\,994.8}{11.5} = -521.3 \text{ lbf/in}^4$$

$$\Delta m = -521.3 - (-965.8) = 444.5 \text{ lbf/in}^4$$

Dividing Eq. (1) by I_{BC} and, at $x = 8.5$ in, adding a step of $-5\,111.8$ lbf/in^3 and a ramp of slope 444.5 lbf/in^4, gives

$$\frac{M}{I} = 1\,448.7x - 2\,414.5\langle x - 8\rangle^1 - 5\,111.8\langle x - 8.5\rangle^0 + 444.5\langle x - 8.5\rangle^1 \qquad (2)$$

Integration gives

$$E\frac{dy}{dx} = 724.35x^2 - 1207.3\langle x - 8\rangle^2 - 5\,111.8\langle x - 8.5\rangle^1 + 222.3\langle x - 8.5\rangle^2 + C_1 \qquad (3)$$

Example 4-7

Integrating again yields

$$Ey = 241.5x^3 - 402.4\langle x-8\rangle^3 - 2\,555.9\langle x-8.5\rangle^2 + 74.08\langle x-8.5\rangle^3 + C_1x + C_2 \tag{4}$$

At $x = 0$, $y = 0$. This gives $C_2 = 0$ (remember, singularity functions do not exist until the argument is positive). At $x = 20$ in, $y = 0$, and

$$0 = 241.5(20)^3 - 402.4(20-8)^3 - 2\,555.9(20-8.5)^2 + 74.08(20-8.5)^3 + C_1(20)$$

Solving, gives $C_1 = -50\,565$ lbf/in^2. Thus, Eq. (4) becomes, with $E = 30(10)^6$ psi,

$$y = \frac{1}{30(10^6)}(241.5x^3 - 402.4\langle x-8\rangle^3 - 2\,555.9\langle x-8.5\rangle^2 + 74.08\langle x-8.5\rangle^3 - 50\,565x) \tag{5}$$

Example 4-7

When using a spreadsheet, program the following equations:

$$y = \frac{1}{30(10^6)}(241.5x^3 - 50\,565x) \qquad 0 \le x \le 8 \text{ in}$$

$$y = \frac{1}{30(10^6)}[241.5x^3 - 402.4(x-8)^3 - 50\,565x] \qquad 8 \le x \le 8.5 \text{ in}$$

$$y = \frac{1}{30(10^6)}[241.5x^3 - 402.4(x-8)^3 - 2\,555.9(x-8.5)^2$$

$$+ 74.08(x-8.5)^3 - 50\,565x] \qquad 8.5 \le x \le 20 \text{ in}$$

Example 4-7

The following table results.

x	y	x	y	x	y	x	y	x	y
0	0.000000	4.5	−0.006851	9	−0.009335	13.5	−0.007001	18	−0.002377
0.5	−0.000842	5	−0.007421	9.5	−0.009238	14	−0.006571	18.5	−0.001790
1	−0.001677	5.5	−0.007931	10	−0.009096	14.5	−0.006116	19	−0.001197
1.5	−0.002501	6	−0.008374	10.5	−0.008909	15	−0.005636	19.5	−0.000600
2	−0.003307	6.5	−0.008745	11	−0.008682	15.5	−0.005134	20	0.000000
2.5	−0.004088	7	−0.009037	11.5	−0.008415	16	−0.004613		
3	−0.004839	7.5	−0.009245	12	−0.008112	16.5	−0.004075		
3.5	−0.005554	8	−0.009362	12.5	−0.007773	17	−0.003521		
4	−0.006227	8.5	−0.009385	13	−0.007403	17.5	−0.002954		

where x and y are in inches. We see that the greatest deflection is at $x = 8.5$ in, where $y = -0.009385$ in.

Example 4-7

Substituting C_1 into Eq. (3) the slopes at the supports are found to be $\theta_A = 1.686(10^{-3})$ rad = 0.09657 deg, and $\theta_F = 1.198(10^{-3})$ rad = 0.06864 deg. You might think these to be insignificant deflections, but as you will see in Chap. 7, on shafts, they are not.

A finite-element analysis was performed for the same model and resulted in

$$y|_{x=8.5\,\text{in}} = -0.009380\,\text{in} \qquad \theta_A = -0.09653^\circ \qquad \theta_F = 0.06868^\circ$$

Virtually the same answer save some round-off error in the equations.

If the steps of the bearings were incorporated into the model, more equations result, but the process is the same. The solution to this model is

$$y|_{x=8.5\,\text{in}} = -0.009387\,\text{in} \qquad \theta_A = -0.09763^\circ \qquad \theta_F = 0.06973^\circ$$

The largest difference between the models is of the order of 1.5 percent. Thus the simplification was justified.

Strain Energy

- External work done on elastic member in deforming it is transformed into *strain energy*, or *potential energy*.
- Strain energy equals product of average force and deflection.

$$U = \frac{F}{2}y = \frac{F^2}{2k} \qquad (4\text{–}15)$$

Some Common Strain Energy Formulas

- For axial loading, applying $k = AE/l$ from Eq. (4-4),

$$U = \frac{F^2 l}{2AE} \tag{4-16}$$

or

$$U = \int \frac{F^2}{2AE}\,dx \tag{4-17}$$

} tension and compression

- For torsional loading, applying $k = GJ/l$ from Eq. (4-7),

$$U = \frac{T^2 l}{2GJ} \tag{4-18}$$

or

$$U = \int \frac{T^2}{2GJ}\,dx \tag{4-19}$$

} torsion

Some Common Strain Energy Formulas

- For direct shear loading,

$$U = \frac{F^2 l}{2AG} \quad (4\text{–}20)$$

or

$$U = \int \frac{F^2}{2AG}\, dx \quad (4\text{–}21)$$

$\Big\}$ direct shear

- For bending loading,

$$U = \frac{M^2 l}{2EI} \quad (4\text{–}22)$$

or

$$U = \int \frac{M^2}{2EI}\, dx \quad (4\text{–}23)$$

$\Big\}$ bending

Some Common Strain Energy Formulas

- For transverse shear loading,

$$U = \frac{CV^2 l}{2AG} \qquad (4\text{–}24)$$

or

$$U = \int \frac{CV^2}{2AG}\,dx \qquad (4\text{–}25)$$

transverse shear

where C is a modifier dependent on the cross sectional shape.

Table 4–1

Strain-Energy Correction Factors for Transverse Shear

Source: Richard G. Budynas, *Advanced Strength and Applied Stress Analysis,* 2nd ed., McGraw-Hill, New York, 1999. Copyright © 1999 The McGraw-Hill Companies.

Beam Cross-Sectional Shape	Factor C
Rectangular	1.2
Circular	1.11
Thin-walled tubular, round	2.00
Box sections†	1.00
Structural sections†	1.00

†Use area of web only.

Summary of Common Strain Energy Formulas

$$\left.\begin{aligned} U &= \frac{F^2 l}{2AE} \\ U &= \int \frac{F^2}{2AE}\,dx \end{aligned}\right\}\ \text{tension and compression}$$

$$\left.\begin{aligned} U &= \frac{T^2 l}{2GJ} \\ U &= \int \frac{T^2}{2GJ}\,dx \end{aligned}\right\}\ \text{torsion}$$

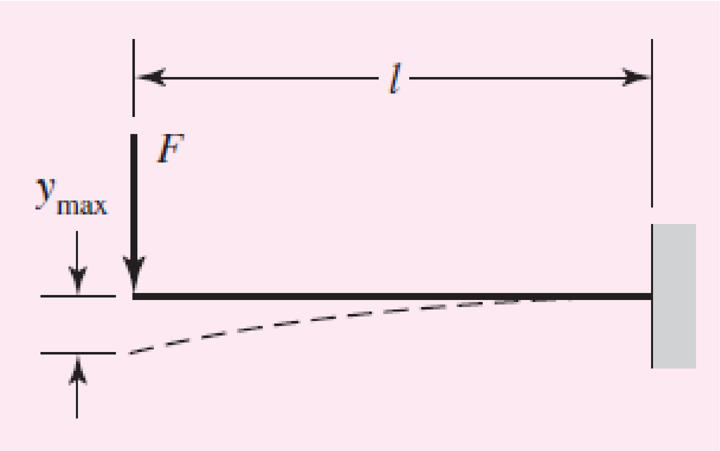
$$\left.\begin{aligned} U &= \frac{F^2 l}{2AG} \\ U &= \int \frac{F^2}{2AG}\,dx \end{aligned}\right\}\ \text{direct shear}$$

$$\left.\begin{aligned} U &= \frac{M^2 l}{2EI} \\ U &= \int \frac{M^2}{2EI}\,dx \end{aligned}\right\}\ \text{bending}$$

$$\left.\begin{aligned} U &= \frac{CV^2 l}{2AG} \\ U &= \int \frac{CV^2}{2AG}\,dx \end{aligned}\right\}\ \text{transverse shear}$$

Example 4-8

A cantilever beam with a round cross section has a concentrated load F at the end, as shown in Fig. 4–9a. Find the strain energy in the beam.



(a)

Fig. 4–9

Example 4-8

To determine what forms of strain energy are involved with the deflection of the beam, we break into the beam and draw a free-body diagram to see the forces and moments being carried within the beam. Figure 4–9*b* shows such a diagram in which the transverse shear is $V = -F$, and the bending moment is $M = -Fx$. The variable x is simply a variable of integration and can be defined to be measured from any convenient point. The same results will be obtained from a free-body diagram of the right-hand portion of the beam with x measured from the wall. Using the free end of the beam usually results in reduced effort since the ground reaction forces do not need to be determined.

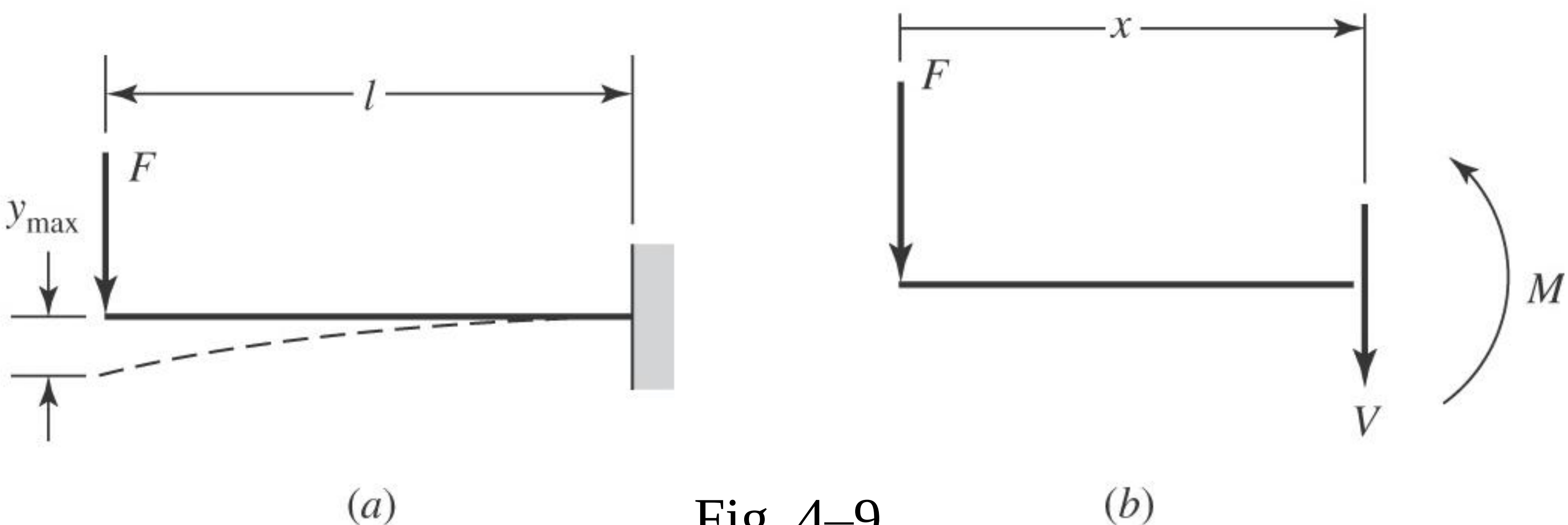


Fig. 4–9

Example 4-8

For the transverse shear, using Eq. (4–24) with the correction factor $C = 1.11$ from Table 4–2, and noting that V is constant through the length of the beam,

$$U_{\text{shear}} = \frac{CV^2 l}{2AG} = \frac{1.11F^2 l}{2AG}$$

For the bending, since M is a function of x, Eq. (4–23) gives

$$U_{\text{bend}} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^l (-Fx)^2 dx = \frac{F^2 l^3}{6EI}$$

The total strain energy is

$$U = U_{\text{bend}} + U_{\text{shear}} = \frac{F^2 l^3}{6EI} + \frac{1.11F^2 l}{2AG}$$

Note, except for very short beams, the shear term (of order l) is typically small compared to the bending term (of order l^3). This will be demonstrated in the next example.

Castigliano's Theorem

- When forces act on elastic systems subject to small displacements, the displacement corresponding to any force, in the direction of the force, is equal to the partial derivative of the total strain energy with respect to that force.

$$\delta_i = \frac{\partial U}{\partial F_i} \qquad (4\text{–}26)$$

- For rotational displacement, in radians,

$$\theta_i = \frac{\partial U}{\partial M_i} \qquad (4\text{–}27)$$

Example 4-9

The cantilever of Ex. 4–8 is a carbon steel bar 10 in long with a 1-in diameter and is loaded by a force $F = 100$ lbf.

(a) Find the maximum deflection using Castigliano's theorem, including that due to shear.

(b) What error is introduced if shear is neglected?

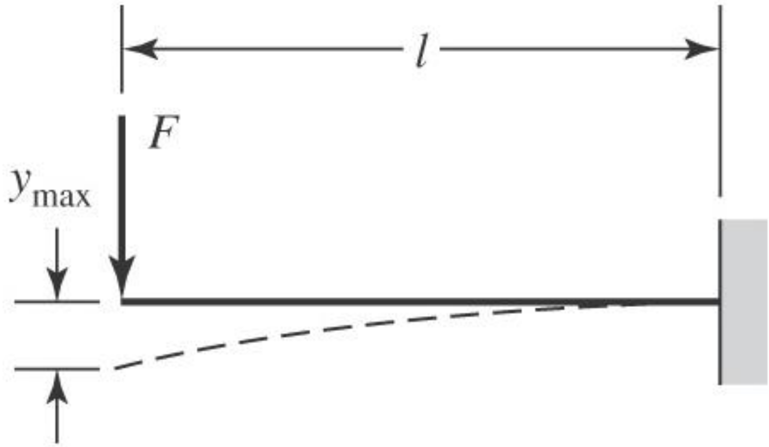


Fig. 4–9

Example 4-9

(*a*) From Ex. 4–8, the total energy of the beam is

$$U = \frac{F^2 l^3}{6EI} + \frac{1.11 F^2 l}{2AG} \qquad (1)$$

Then, according to Castigliano's theorem, the deflection of the end is

$$y_{max} = \frac{\partial U}{\partial F} = \frac{F l^3}{3EI} + \frac{1.11 F l}{AG} \qquad (2)$$

We also find that

$$I = \frac{\pi d^4}{64} = \frac{\pi (1)^4}{64} = 0.0491 \text{ in}^4$$

$$A = \frac{\pi d^2}{4} = \frac{\pi (1)^2}{4} = 0.7854 \text{ in}^2$$

Substituting these values, together with $F = 100$ lbf, $l = 10$ in, $E = 30$ Mpsi, and $G = 11.5$ Mpsi, in Eq. (3) gives

$$y_{max} = 0.022\,63 + 0.000\,12 = 0.022\,75 \text{ in}$$

Note that the result is positive because it is in the *same* direction as the force F.

(*b*) The error in neglecting shear for this problem is $(0.02275 - 0.02263)/0.02275 = 0.0053 = 0.53$ percent.

Utilizing a Fictitious Force

- Castigliano's method can be used to find a deflection at a point even if there is no force applied at that point.
- Apply a fictitious force Q at the point, and in the direction, of the desired deflection.
- Set up the equation for total strain energy including the energy due to Q.
- Take the derivative of the total strain energy with respect to Q.
- Once the derivative is taken, Q is no longer needed and can be set to zero.

$$\delta = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} \qquad (4\text{–}28)$$

Finding Deflection Without Finding Energy

- For cases requiring integration of strain energy equations, it is more efficient to obtain the deflection directly without explicitly finding the strain energy.
- The partial derivative is moved inside the integral.
- For example, for bending,

$$\delta_i = \frac{\partial}{\partial F_i} = \frac{\partial}{\partial F_i}\left(\int \frac{M^2}{2EI}\,dx\right) = \int \frac{\partial}{\partial F_i}\left(\frac{M^2}{2EI}\right) dx = \int \frac{2M\dfrac{\partial M}{\partial F_i}}{2EI}\,dx$$

$$= \int \frac{1}{EI}\left(M\frac{\partial M}{\partial F_i}\right) dx$$

- Derivative can be taken before integration, simplifying the math.
- Especially helpful with fictitious force *Q*, since it can be set to zero after the derivative is taken.

Common Deflection Equations

$$\delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{AE}\left(F\frac{\partial F}{\partial F_i}\right)dx \qquad \text{tension and compression} \qquad (4\text{–}29)$$

$$\theta_i = \frac{\partial U}{\partial M_i} = \int \frac{1}{GJ}\left(T\frac{\partial T}{\partial M_i}\right)dx \qquad \text{torsion} \qquad (4\text{–}30)$$

$$\delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{EI}\left(M\frac{\partial M}{\partial F_i}\right)dx \qquad \text{bending} \qquad (4\text{–}31)$$

Example 4-10

Using Castigliano's method, determine the deflections of points A and B due to the force F applied at the end of the step shaft shown in Fig. 4–10. The second area moments for sections AB and BC are I_1 and $2I_1$, respectively.

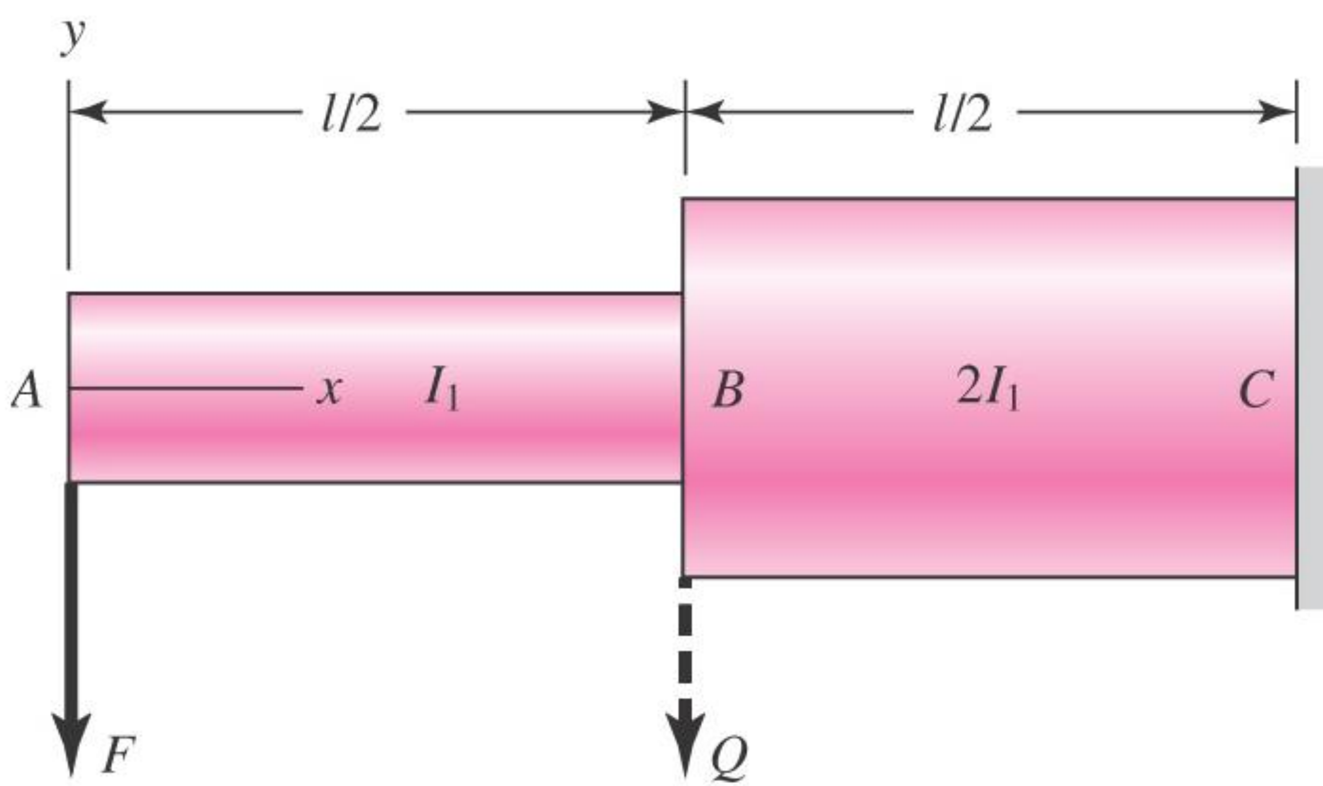


Fig. 4–10

Example 4-10

To avoid the need to determine the ground reaction forces, define the origin of x at the left end of the beam as shown. For $0 \le x \le l$, the bending moment is

$$M = -Fx \tag{1}$$

Since F is at A and in the direction of the desired deflection, the deflection at A from Eq. (4–31) is

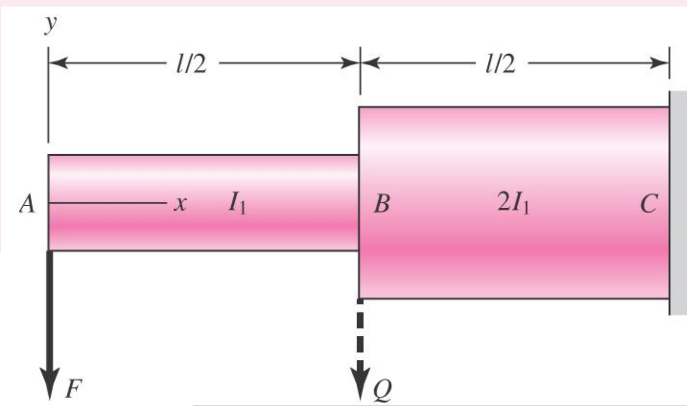
$$\delta_A = \frac{\partial U}{\partial F} = \int_0^l \frac{1}{EI}\left(M\frac{\partial M}{\partial F}\right) dx \tag{2}$$

Substituting Eq. (1) into Eq. (2), noting that $I = I_1$ for $0 \le x \le l/2$, and $I = 2I_1$ for $l/2 \le x \le l$, we get

$$\delta_A = \frac{1}{E}\left[\int_0^{l/2} \frac{1}{I_1}(-Fx)(-x)\,dx + \int_{l/2}^{l} \frac{1}{2I_1}(-Fx)(-x)\,dx\right]$$

$$= \frac{1}{E}\left[\frac{Fl^3}{24I_1} + \frac{7Fl^3}{48I_1}\right] = \frac{3}{16}\frac{Fl^3}{EI_1}$$

which is positive, as it is in the direction of F.



Example 4-10

For B, a fictitious force Q is necessary at the point. Assuming Q acts down at B, and x is as before, the moment equation is

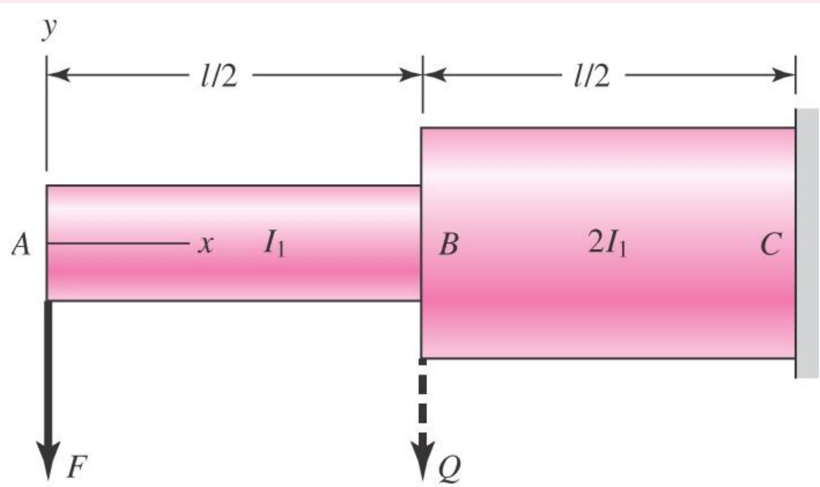
$$M = -Fx \qquad 0 \le x \le l/2$$

$$M = -Fx - Q\left(x - \frac{l}{2}\right) \qquad l/2 \le x \le l \tag{3}$$

For Eq. (4–31), we need $\partial M/\partial Q$. From Eq. (3),

$$\frac{\partial M}{\partial Q} = 0 \qquad 0 \le x \le l/2$$

$$\frac{\partial M}{\partial Q} = -\left(x - \frac{l}{2}\right) \qquad l/2 \le x \le l \tag{4}$$



Example 4-10

Once the derivative is taken, Q can be set to zero, so Eq. (4–31) becomes

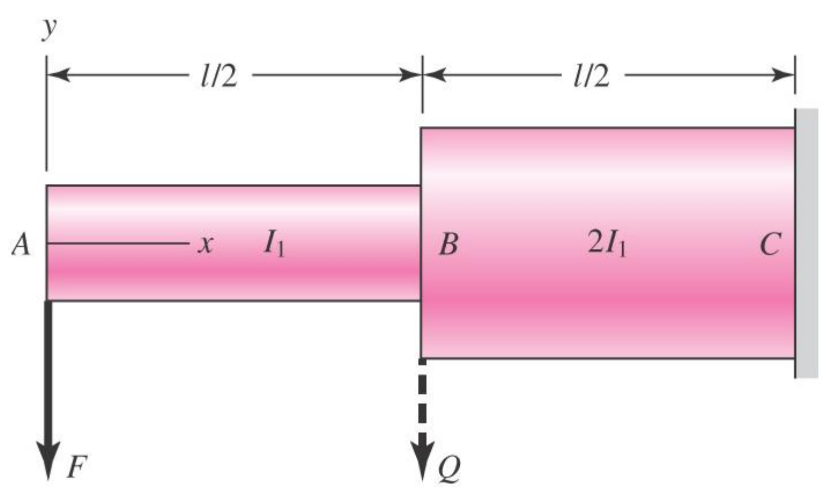
$$\delta_B = \left[\int_0^l \frac{1}{EI}\left(M\frac{\partial M}{\partial Q}\right)dx\right]_{Q=0}$$

$$= \frac{1}{EI_1}\int_0^{l/2}(-Fx)(0)dx + \frac{1}{E(2I_1)}\int_{l/2}^{l}(-Fx)\left[-\left(x-\frac{l}{2}\right)\right]dx$$

Evaluating the last integral gives

$$\delta_B = \frac{F}{2EI_1}\left(\frac{x^3}{3}-\frac{lx^2}{4}\right)\Bigg|_{l/2}^{l} = \frac{5}{96}\frac{Fl^3}{EI_1}$$

which again is positive, in the direction of Q.



Example 4-11

For the wire form of diameter d shown in Fig. 4–11a, determine the deflection of point B in the direction of the applied force F (neglect the effect of transverse shear).

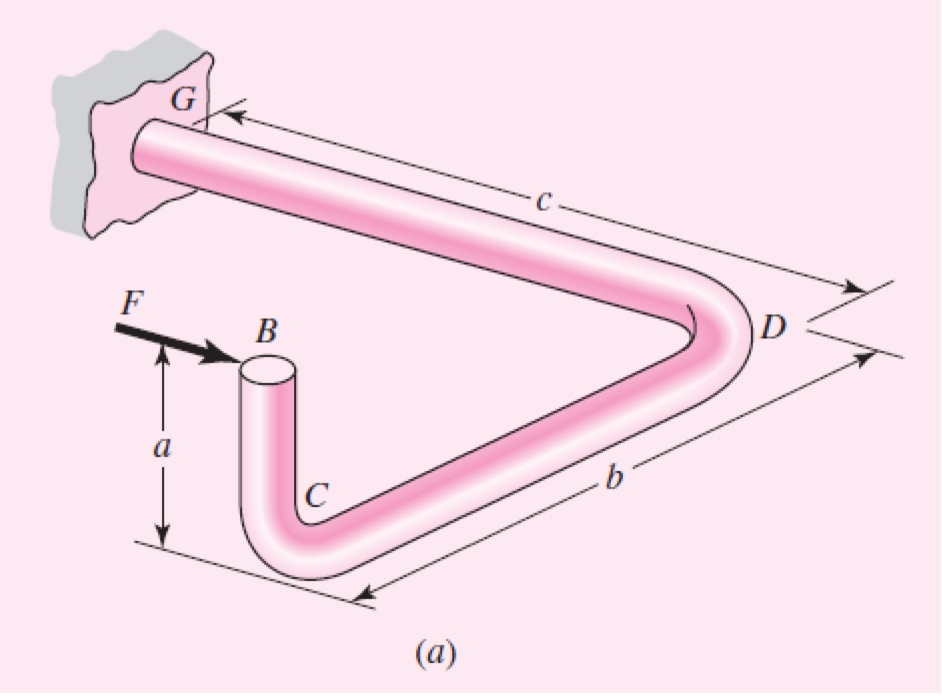
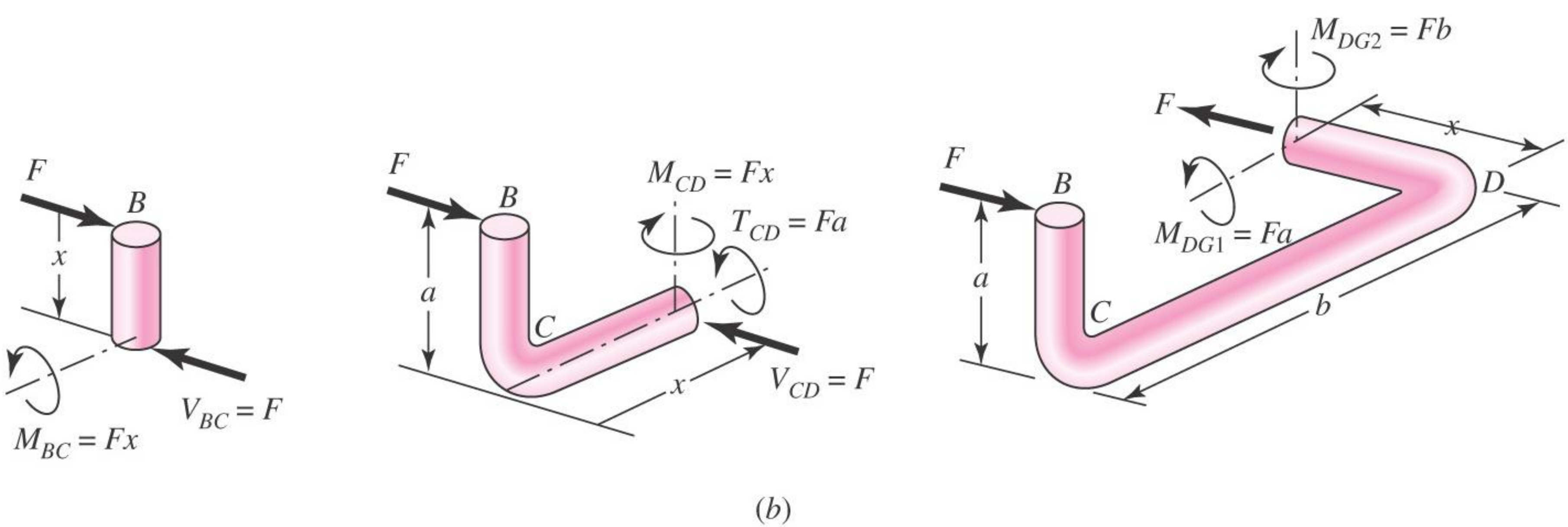


Fig. 4–11

Example 4-11

Figure 4-11*b* shows free body diagrams where the body has been broken in each section, and internal balancing forces and moments are shown. The sign convention for the force and moment variables is positive in the directions shown. With energy methods, sign conventions are arbitrary, so use a convenient one. In each section, the variable x is defined with its origin as shown. The variable x is used as a variable of integration for each section independently, so it is acceptable to reuse the same variable for each section. For completeness, the transverse shear forces are included, but the effects of transverse shear on the strain energy (and deflection) will be neglected.



(*b*)

Fig. 4–11

Example 4-11

Element BC is in bending only so from Eq. (4–31),[5]

$$\frac{\partial U_{BC}}{\partial F} = \frac{1}{EI}\int_0^a (Fx)(x)\,dx = \frac{Fa^3}{3EI} \qquad (1)$$

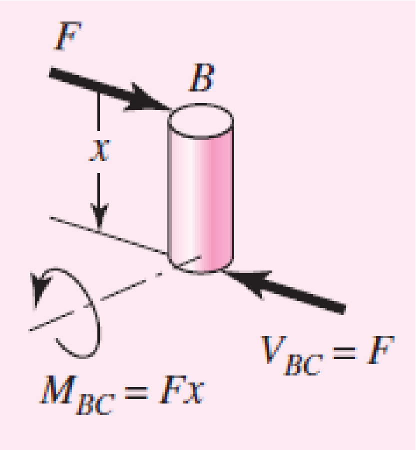


Fig. 4–11

Example 4-11

Element CD is in bending and in torsion. The torsion is constant so Eq. (4–30) can be written as

$$\frac{\partial U}{\partial F_i} = \left(T\frac{\partial T}{\partial F_i}\right)\frac{l}{GJ}$$

where l is the length of the member. So for the torsion in member CD, $F_i = F$, $T = Fa$, and $l = b$. Thus,

$$\left(\frac{\partial U_{CD}}{\partial F}\right)_{\text{torsion}} = (Fa)(a)\frac{b}{GJ} = \frac{Fa^2 b}{GJ} \qquad (2)$$

For the bending in CD,

$$\left(\frac{\partial U_{CD}}{\partial F}\right)_{\text{bending}} = \frac{1}{EI}\int_0^b (Fx)(x)\,dx = \frac{Fb^3}{3EI} \qquad (3)$$

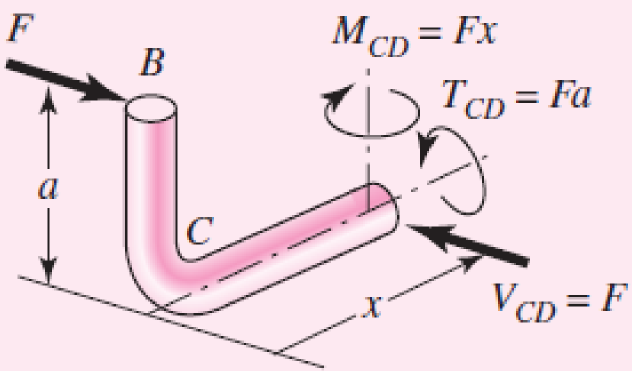


Fig. 4–11

Example 4-11

Member DG is axially loaded and is bending in two planes. The axial loading is constant, so Eq. (4–29) can be written as

$$\frac{\partial U}{\partial F_i} = \left(F \frac{\partial F}{\partial F_i}\right) \frac{l}{AE}$$

where l is the length of the member. Thus, for the axial loading of DG, $F_i = F$, $l = c$, and

$$\left(\frac{\partial U_{DG}}{\partial F}\right)_{\text{axial}} = \frac{Fc}{AE} \tag{4}$$

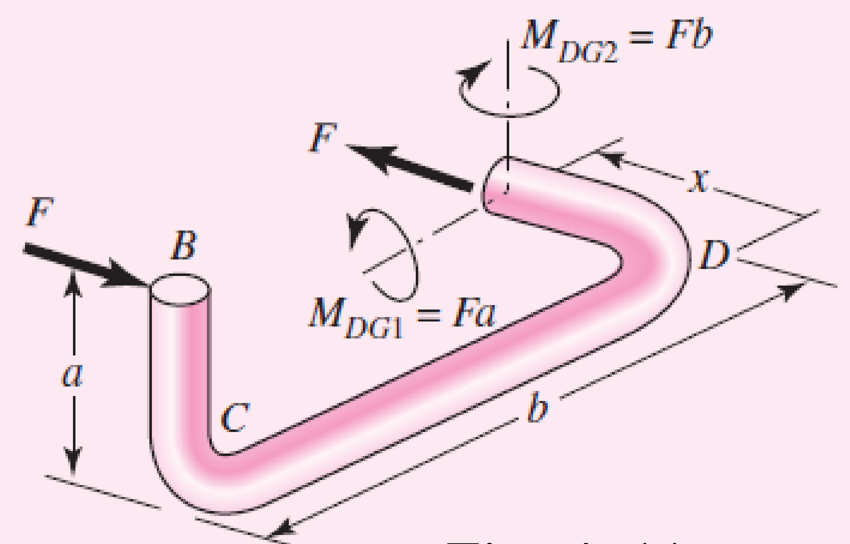


Fig. 4–11

Example 4-11

The bending moments in each plane of DG are constant along the length, with $M_{DG2} = Fb$ and $M_{DG1} = Fa$. Considering each one separately in the form of Eq. (4–31) gives

$$\left(\frac{\partial U_{DG}}{\partial F}\right)_{\text{bending}} = \frac{1}{EI}\int_0^c (Fb)(b)\,dx + \frac{1}{EI}\int_0^c (Fa)(a)\,dx \quad (5)$$

$$= \frac{Fc(a^2 + b^2)}{EI}$$

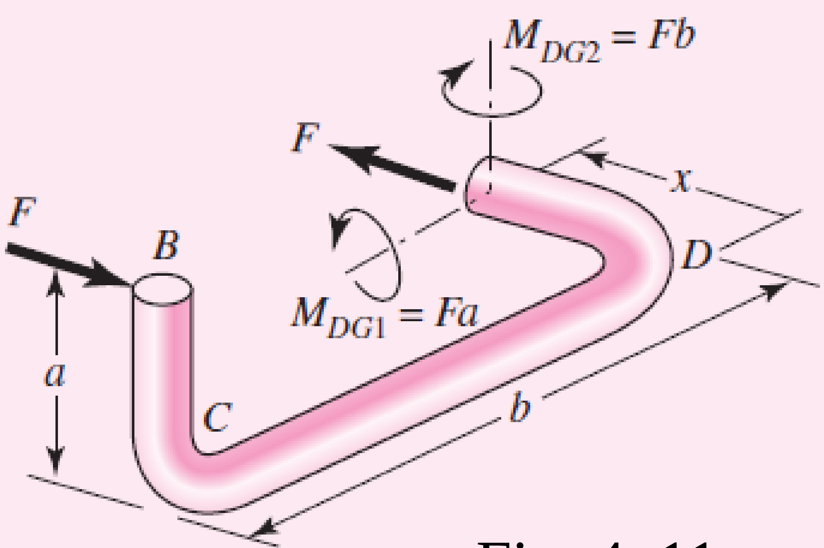


Fig. 4–11

Example 4-11

Adding Eqs. (1) to (5), noting that $I = \pi d^4/64$, $J = 2I$, $A = \pi d^2/4$, and $G = E/[2(1+\nu)]$, we find that the deflection of B in the direction of F is

$$(\delta_B)_F = \frac{4F}{3\pi E d^4}[16(a^3+b^3) + 48c(a^2+b^2) + 48(1+\nu)a^2 b + 3cd^2]$$

Now that we have completed the solution, see if you can physically account for each term in the result using an independent method such as superposition.

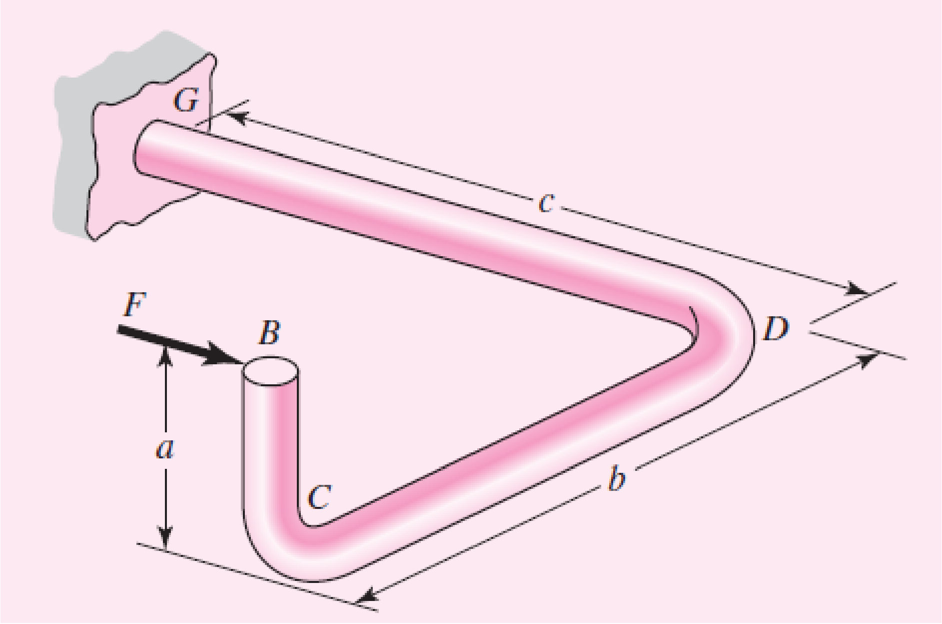


Fig. 4–11

Deflection of Curved Members

- Consider case of thick curved member in bending (See Sec. 3-18)
- Four strain energy terms due to
  - Bending moment M
  - Axial force F_θ
  - Bending moment due to F_θ (since neutral axis and centroidal axis do not coincide)
  - Transverse shear F_r

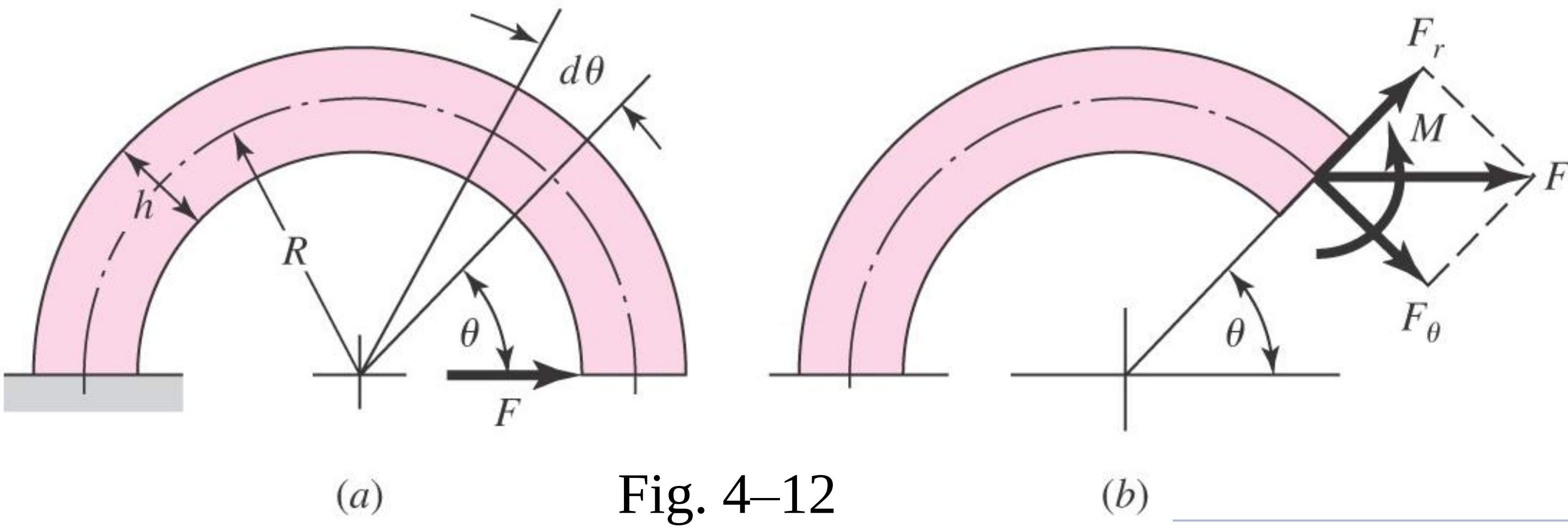


Fig. 4–12

Deflection of Curved Members

- Strain energy due to bending moment M

$$U_1 = \int \frac{M^2 \, d\theta}{2AeE} \quad (4\text{–}32)$$

$$e = R - r_n \quad (4\text{–}33)$$

where r_n is the radius of the neutral axis

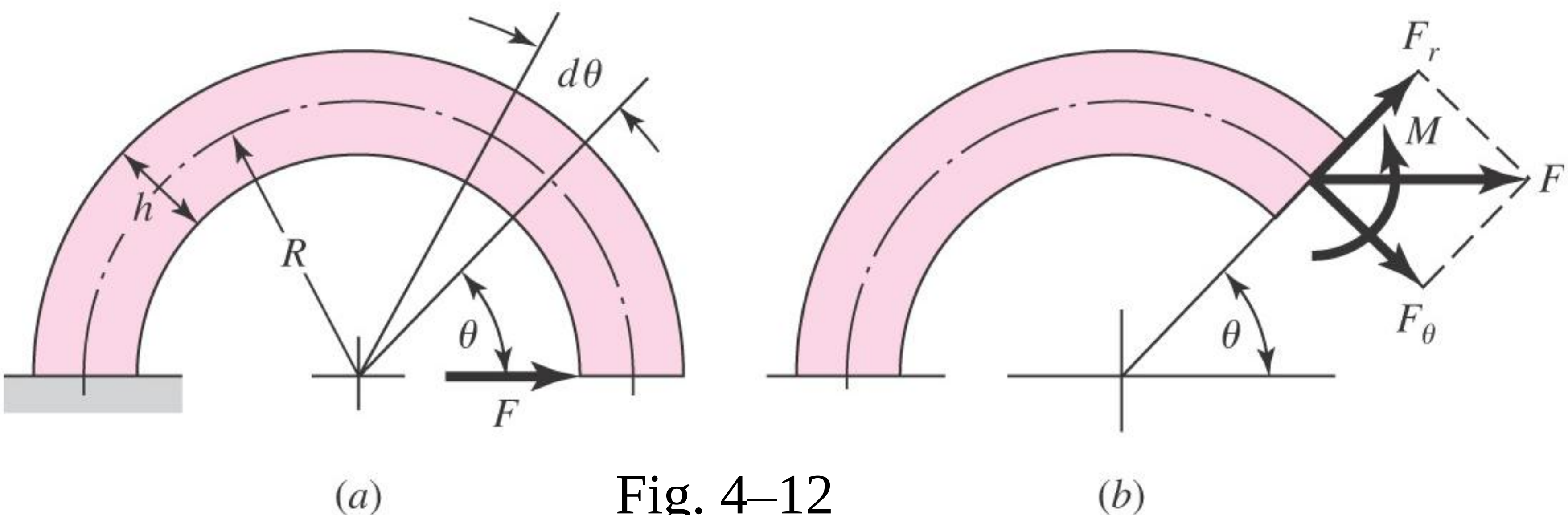


Fig. 4–12

Deflection of Curved Members

- Strain energy due to axial force F_θ

$$U_2 = \int \frac{F_\theta^2 R \, d\theta}{2AE} \qquad \text{(4–34)}$$

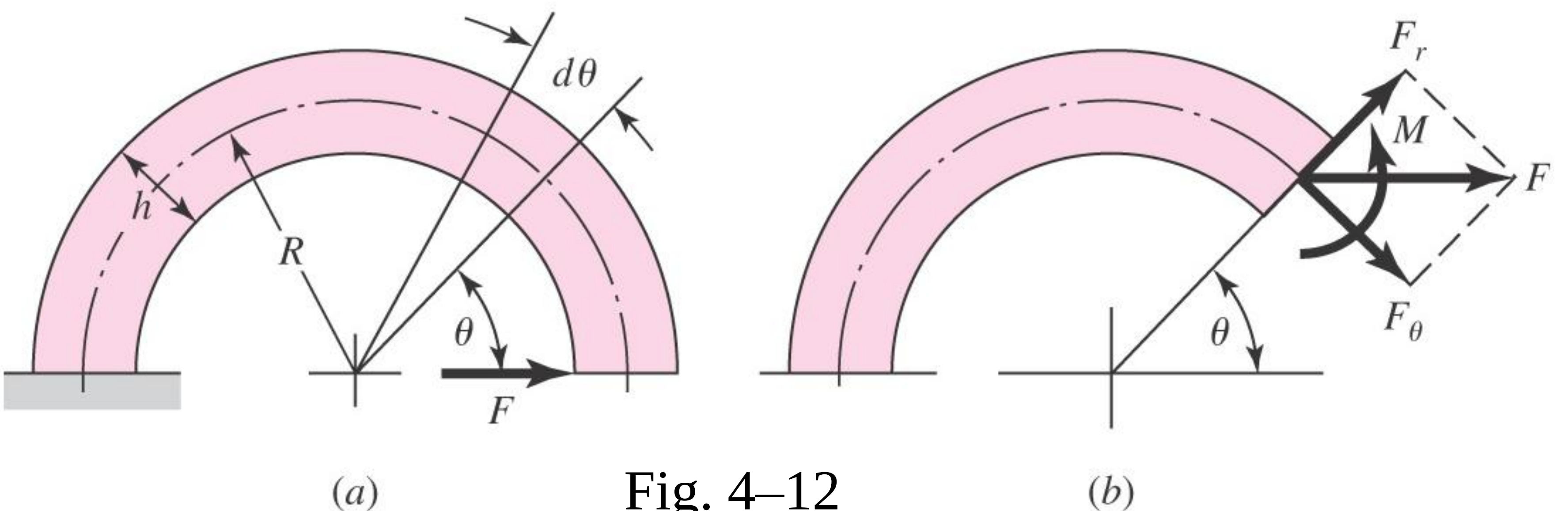


Fig. 4–12

Deflection of Curved Members

- Strain energy due to bending moment due to F_θ

$$U_3 = -\int \frac{M F_\theta \, d\theta}{AE} \qquad (4\text{–}35)$$

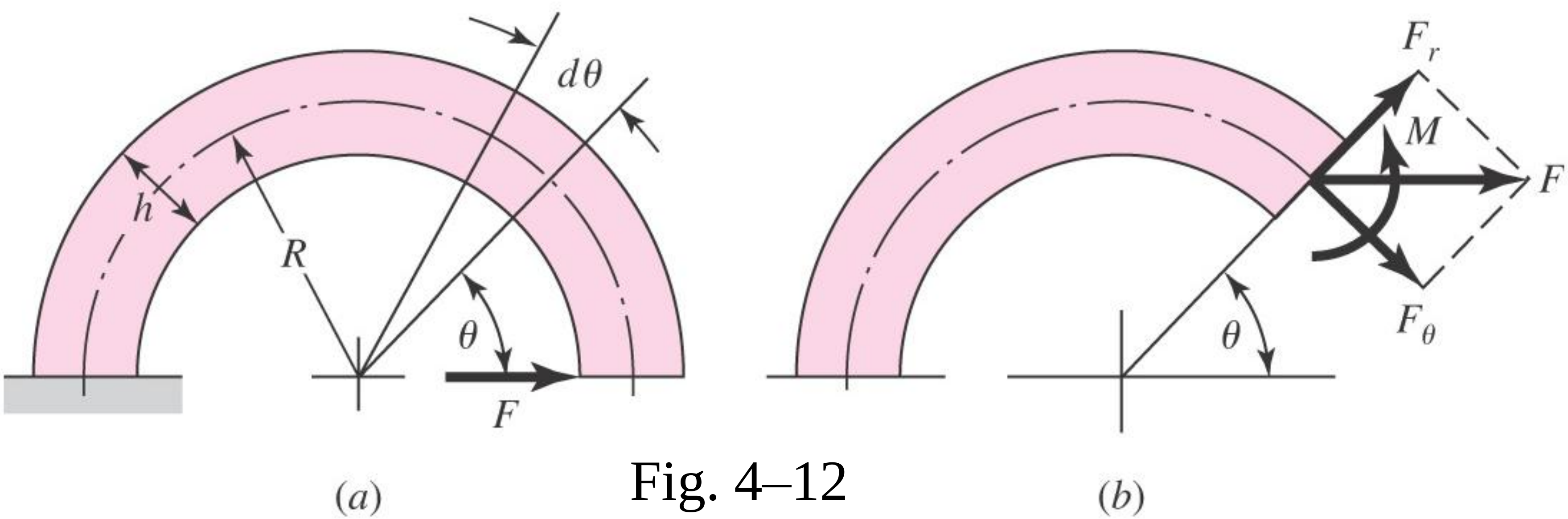


Fig. 4–12

Deflection of Curved Members

- Strain energy due to transverse shear F_r

$$U_4 = \int \frac{C F_r^2 R\, d\theta}{2AG} \qquad (4\text{–}36)$$

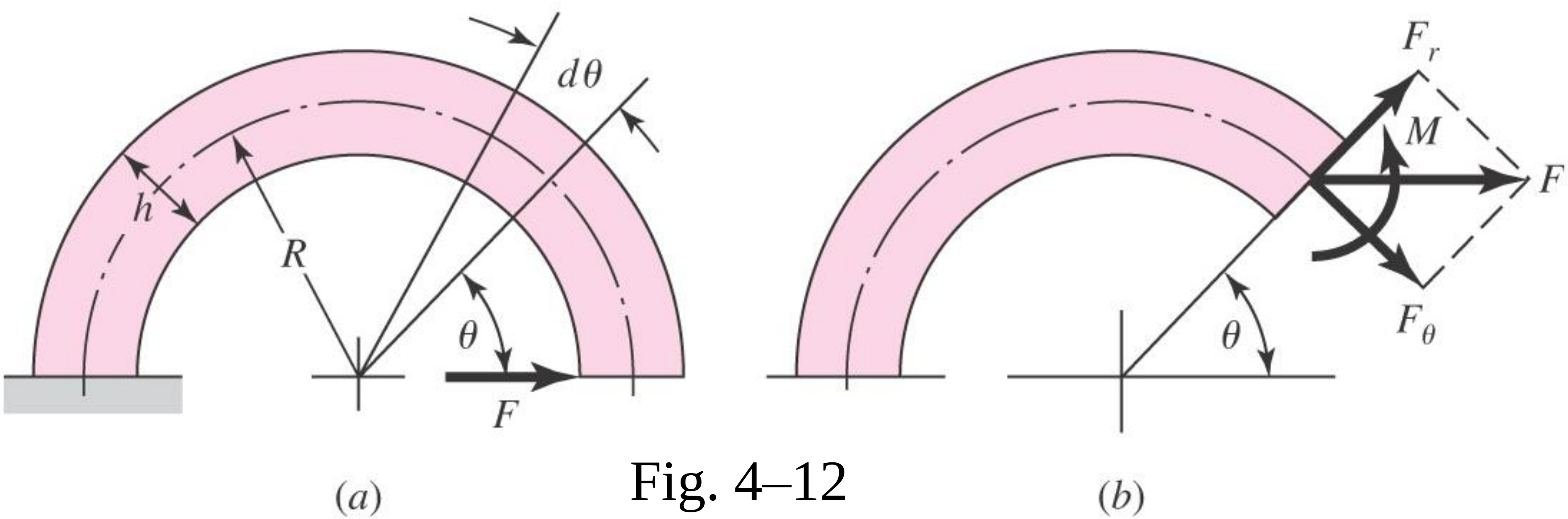


Fig. 4–12

Deflection of Curved Members

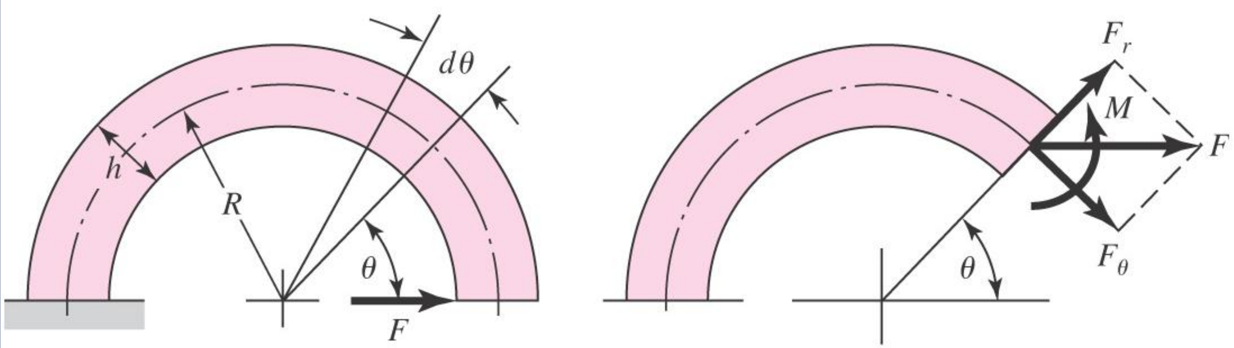
- Combining four energy terms

$$U = \int \frac{M^2\,d\theta}{2AeE} + \int \frac{F_\theta^2 R\,d\theta}{2AE} - \int \frac{MF_\theta\,d\theta}{AE} + \int \frac{CF_r^2 R\,d\theta}{2AG} \qquad (4\text{–}37)$$

- Deflection by Castigliano's method

$$\delta = \frac{\partial U}{\partial F} = \int \frac{M}{AeE}\left(\frac{\partial M}{\partial F}\right) d\theta + \int \frac{F_\theta R}{AE}\left(\frac{\partial F_\theta}{\partial F}\right) d\theta - \int \frac{1}{AE}\frac{\partial (MF_\theta)}{\partial F} d\theta + \int \frac{CF_r R}{AG}\left(\frac{\partial F_r}{\partial F}\right) d\theta \qquad (4\text{–}38)$$

- General for any thick circular curved member, with appropriate limits of integration



Deflection of Curved Members

- For specific example in figure,

$$M = FR\sin\theta \qquad \frac{\partial M}{\partial F} = R\sin\theta$$

$$F_\theta = F\sin\theta \qquad \frac{\partial F_\theta}{\partial F} = \sin\theta$$

$$MF_\theta = F^2 R\sin^2\theta \qquad \frac{\partial (MF_\theta)}{\partial F} = 2FR\sin^2\theta$$

$$F_r = F\cos\theta \qquad \frac{\partial F_r}{\partial F} = \cos\theta$$

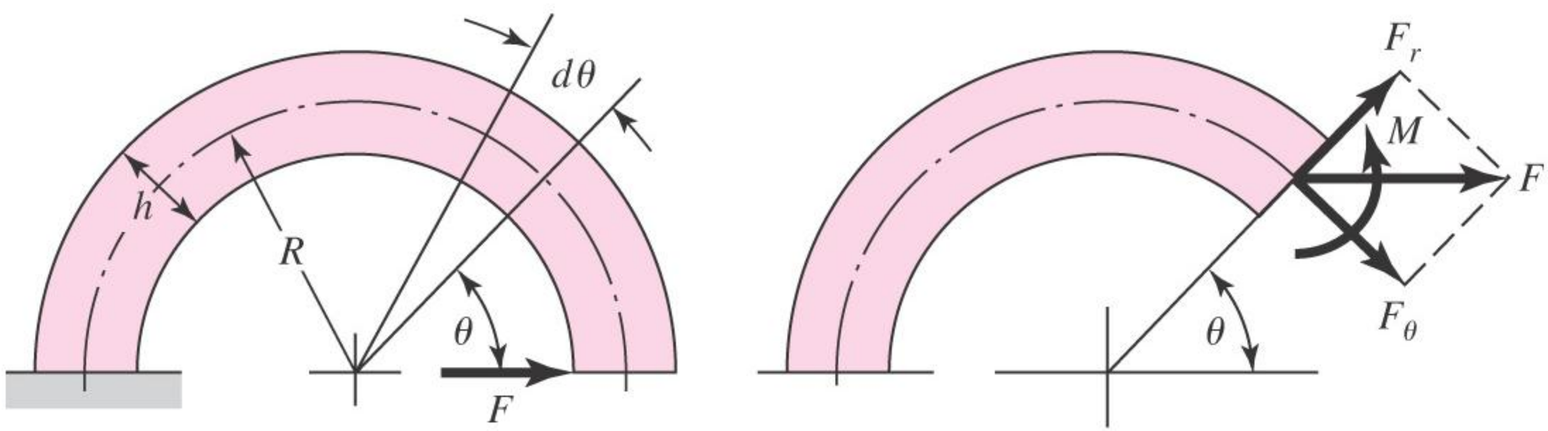


Fig. 4–12

Deflection of Curved Members

- Substituting and factoring,

$$\delta = \frac{FR^2}{AeE}\int_0^\pi \sin^2\theta\,d\theta + \frac{FR}{AE}\int_0^\pi \sin^2\theta\,d\theta - \frac{2FR}{AE}\int_0^\pi \sin^2\theta\,d\theta + \frac{CFR}{AG}\int_0^\pi \cos^2\theta\,d\theta$$

$$= \frac{\pi FR^2}{2AeE} + \frac{\pi FR}{2AE} - \frac{\pi FR}{AE} + \frac{\pi CFR}{2AG} = \frac{\pi FR^2}{2AeE} - \frac{\pi FR}{2AE} + \frac{\pi CFR}{2AG} \qquad (4\text{–}39)$$

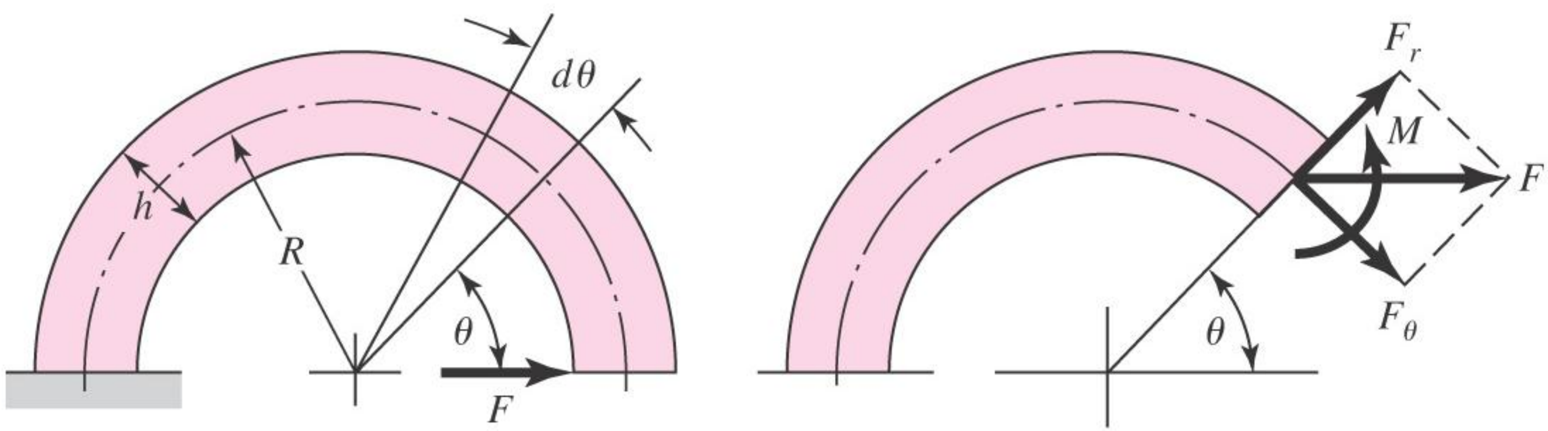


Fig. 4–12

Deflection of Thin Curved Members

- For *thin* curved members, say $R/h > 10$, eccentricity is small
- Strain energies can be approximated with regular energy equations with substitution of $Rd\theta$ for dx
- As R increases, bending component dominates all other terms

$$U \doteq \int \frac{M^2}{2EI} R\, d\theta \qquad R/h > 10 \qquad (4\text{–}40)$$

$$\delta = \frac{\partial U}{\partial F} \doteq \int \frac{1}{EI}\left(M \frac{\partial M}{\partial F}\right) R\, d\theta \qquad R/h > 10 \qquad (4\text{–}41)$$

Example 4-12

The cantilevered hook shown in Fig. 4–13*a* is formed from a round steel wire with a diameter of 2 mm. The hook dimensions are $l = 40$ and $R = 50$ mm. A force P of 1 N is applied at point C. Use Castigliano's theorem to estimate the deflection at point D at the tip.

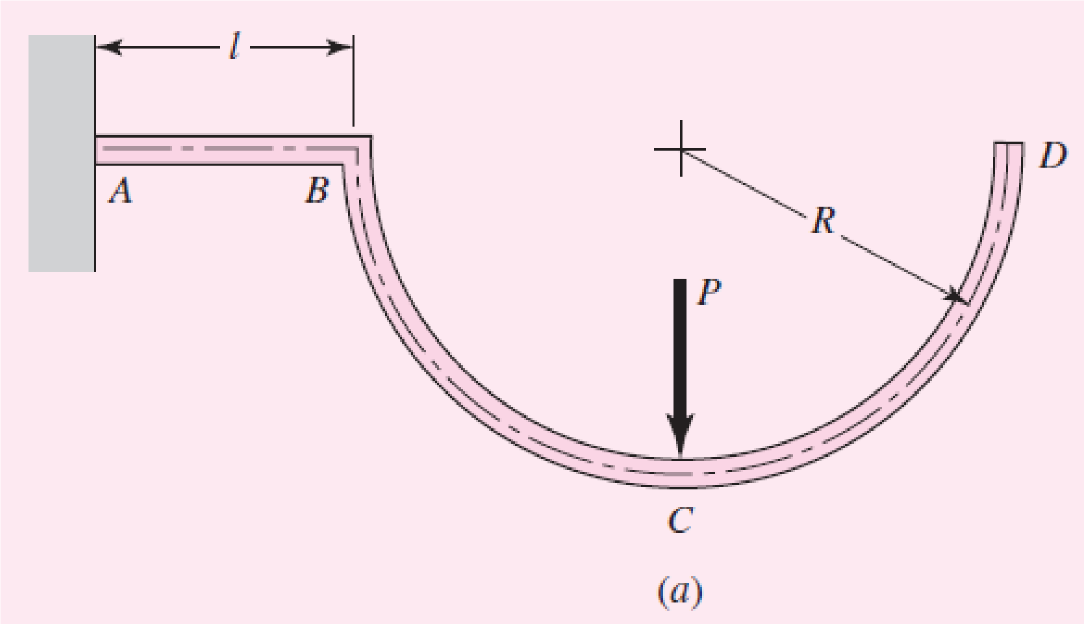


Fig. 4–13

Example 4-12

Since l/d and R/d are significantly greater than 10, only the contributions due to bending will be considered. To obtain the vertical deflection at D, a fictitious force Q will be applied there. Free-body diagrams are shown in Figs. 4–13*b*, *c*, and *d*, with breaks in sections AB, BC, and CD, respectively. The normal and shear forces, N and V respectively, are shown but are considered negligible in the deflection analysis.

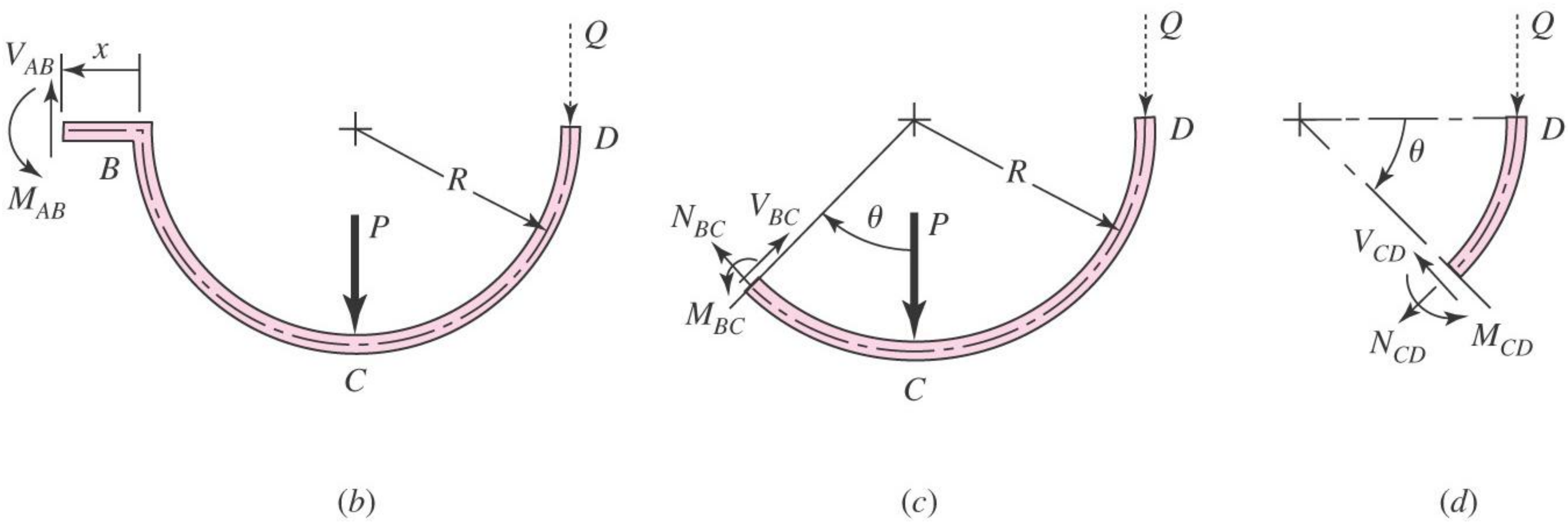


Fig. 4–13

Example 4-12

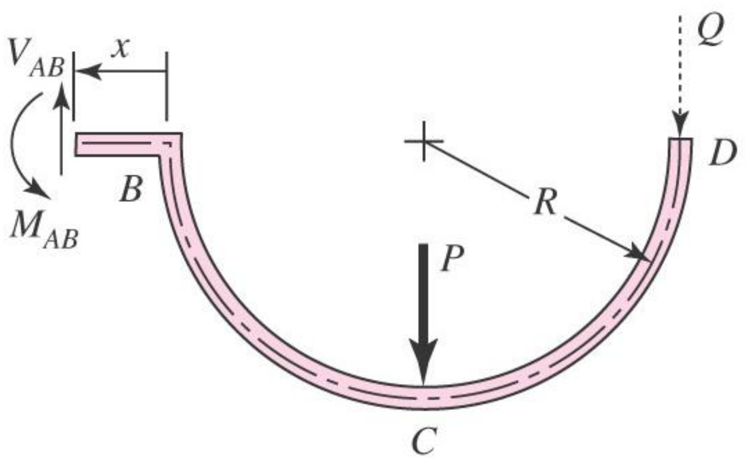
For section AB, with the variable of integration x defined as shown in Fig. 4–13b, summing moments about the break gives an equation for the moment in section AB,

$$M_{AB} = P(R + x) + Q(2R + x) \tag{1}$$

$$\partial M_{AB}/\partial Q = 2R + x \tag{2}$$

Since the derivative with respect to Q has been taken, we can set Q equal to zero. From Eq. (4–31), inserting Eqs. (1) and (2),

$$(\delta_D)_{AB} = \int_0^l \frac{1}{EI}\left(M_{AB}\frac{\partial M_{AB}}{\partial Q}\right)dx = \frac{1}{EI}\int_0^l P(R + x)(2R + x)\,dx$$
$$= \frac{P}{EI}\int_0^l (2R^2 + 3Rx + x^2)dx = \frac{P}{EI}(2R^2 l + \frac{3}{2}l^2 R + \frac{1}{3}l^3) \tag{3}$$



Example 4-12

For section BC, with the variable of integration θ defined as shown in Fig. 4–13c, summing moments about the break gives the moment equation for section BC.

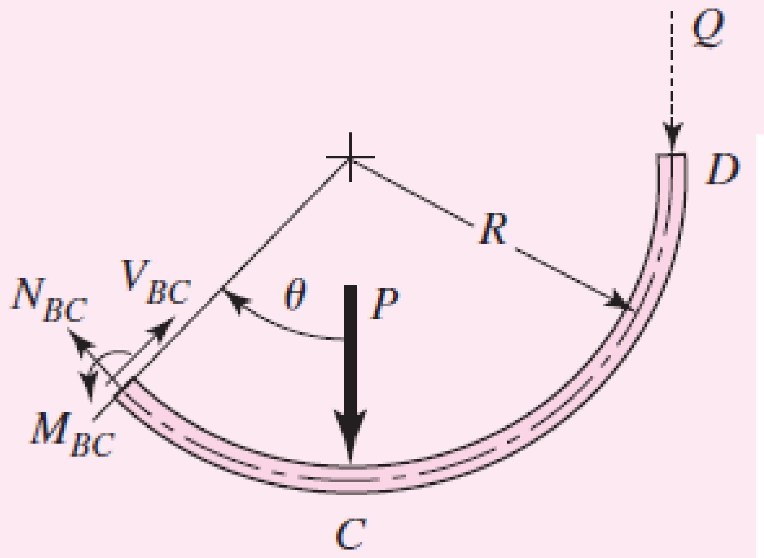
$$M_{BC} = Q(R + R\sin\theta) + PR\sin\theta \tag{4}$$

$$\partial M_{BC}/\partial Q = R(1 + \sin\theta) \tag{5}$$

From Eq. (4–41), inserting Eqs. (4) and (5) and setting $Q = 0$, we get

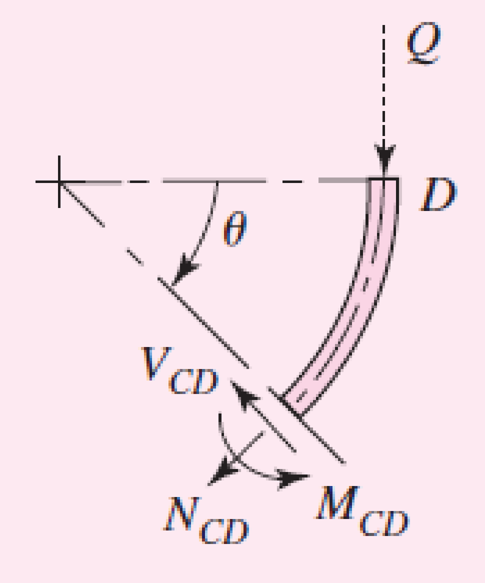
$$(\delta_D)_{BC} = \int_0^{\pi/2} \frac{1}{EI}\left(M_{BC}\frac{\partial M_{BC}}{\partial Q}\right) R\,d\theta = \frac{R}{EI}\int_0^{\pi/2} (PR\sin\theta)[R(1+\sin\theta)]\,dx$$

$$= \frac{PR^3}{EI}\left(1 + \frac{\pi}{4}\right) \tag{6}$$



Example 4-12

Noting that the break in section CD contains nothing but Q, and after setting $Q = 0$, we can conclude that there is no actual strain energy contribution in this section.



Example 4-12

Combining terms from Eqs. (3) and (6) to get the total vertical deflection at D,

$$\delta_D = (\delta_D)_{AB} + (\delta_D)_{BC} = \frac{P}{EI}\left(2R^2 l + \frac{3}{2}l^2 R + \frac{1}{3}l^3\right) + \frac{PR^3}{EI}\left(1 + \frac{\pi}{4}\right)$$

$$= \frac{P}{EI}(1.785R^3 + 2R^2 l + 1.5\,Rl^2 + 0.333l^3) \qquad (7)$$

Substituting values, and noting $I = \pi d^4/64$, and $E = 207$ GPa for steel, we get

$$\delta_D = \frac{1}{207(10^9)[\pi(0.002^4)/64]}[1.785(0.05^3) + 2(0.05^2)0.04 + 1.5(0.05)0.04^2 + 0.333(0.04^3)]$$

$$= 3.47(10^{-3})\text{ m} = 3.47\text{ mm}$$

Example 4-13

Deflection in a Variable-Cross-Section Punch-Press Frame

The general result expressed in Eq. (4–39),

$$\delta = \frac{\pi F R^2}{2AeE} - \frac{\pi F R}{2AE} + \frac{\pi C F R}{2AG}$$

is useful in sections that are uniform and in which the centroidal locus is circular. The bending moment is largest where the material is farthest from the load axis. Strengthening requires a larger second area moment I. A variable-depth cross section is attractive, but it makes the integration to a closed form very difficult. However, if you are seeking results, numerical integration with computer assistance is helpful.

Consider the steel C frame depicted in Fig. 4–14*a* in which the centroidal radius is 32 in, the cross section at the ends is 2 in × 2 in, and the depth varies sinusoidally with

Example 4-13

Consider the steel C frame depicted in Fig. 4–14*a* in which the centroidal radius is 32 in, the cross section at the ends is 2 in × 2 in, and the depth varies sinusoidally with an amplitude of 2 in. The load is 1000 lbf. It follows that $C = 1.2$, $G = 11.5(10^6)$ psi, $E = 30(10^6)$ psi. The outer and inner radii are

$$R_{\text{out}} = 33 + 2\sin\theta \qquad R_{\text{in}} = 31 - 2\sin\theta$$

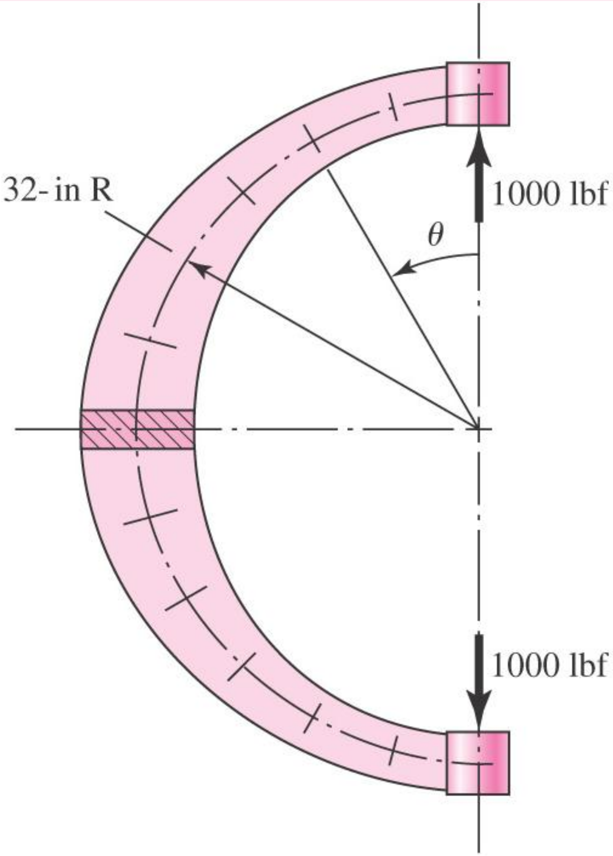


Fig. 4–14

Example 4-13

The remaining geometrical terms are

$$h = R_{\text{out}} - R_{\text{in}} = 2(1 + 2\ \sin\theta)$$

$$A = bh = 4(1 + 2\ \sin\theta)$$

$$r_n = \frac{h}{\ln(R_{\text{out}}/R_{\text{in}})} = \frac{2(1 + 2\ \sin\theta)}{\ln[(33 + 2\ \sin\theta)/(31 - 2\ \sin\theta)]}$$

$$e = R - r_n = 32 - r_n$$

Note that

$$M = FR\sin\theta \qquad \partial M/\partial F = R\sin\theta$$

$$F_\theta = F\sin\theta \qquad \partial F_\theta/\partial F = \sin\theta$$

$$MF_\theta = F^2R\sin^2\theta \qquad \partial MF_\theta/\partial F = 2FR\sin^2\theta$$

$$F_r = F\cos\theta \qquad \partial F_r/\partial F = \cos\theta$$

Example 4-13

Substitution of the terms into Eq. (4–38) yields three integrals

$$\delta = I_1 + I_2 + I_3 \qquad (1)$$

where the integrals are

$$I_1 = 8.5333(10^{-3}) \int_0^{\pi} \frac{\sin^2\theta\, d\theta}{(1 + 2\sin\theta)\left[32 - \dfrac{2(1+2\sin\theta)}{\ln\left(\dfrac{33+2\sin\theta}{31-2\sin\theta}\right)}\right]} \qquad (2)$$

$$I_2 = -2.6667(10^{-4}) \int_0^{\pi} \frac{\sin^2\theta\, d\theta}{1 + 2\sin\theta} \qquad (3)$$

$$I_3 = 8.3478(10^{-4}) \int_0^{\pi} \frac{\cos^2\theta\, d\theta}{1 + 2\sin\theta} \qquad (4)$$

Example 4-13

The integrals may be evaluated in a number of ways: by a program using Simpson's rule integration,[8] by a program using a spreadsheet, or by mathematics software. Using MathCad and checking the results with Excel gives the integrals as $I_1 = 0.076\,615$, $I_2 = -0.000\,159$, and $I_3 = 0.000\,773$. Substituting these into Eq. (1) gives

$$\delta = 0.077\,23 \text{ in}$$

Finite-element (FE) programs are also very accessible. Figure 4–14*b* shows a simple half-model, using symmetry, of the press consisting of 216 plane-stress (2-D) elements. Creating the model and analyzing it to obtain a solution took minutes. Doubling the results from the FE analysis yielded $\delta = 0.07790$ in, a less than 1 percent variation from the results of the numerical integration.

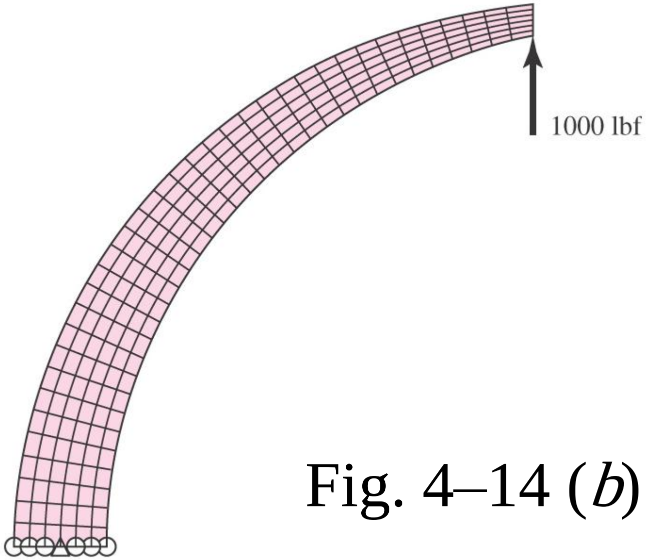


Fig. 4–14 (*b*)

Statically Indeterminate Problems

- A system is *overconstrained* when it has more unknown support (reaction) forces and/or moments than static equilibrium equations.
- Such a system is said to be *statically indeterminate.*
- The extra constraint supports are call *redundant supports.*
- To solve, a deflection equation is required for each redundant support.

Statically Indeterminate Problems

- Example of nested springs
- One equation of static equilibrium

$$\sum F = F - F_1 - F_2 = 0$$

- Deformation equation

$$\delta_1 = \delta_2 = \delta$$

- Use spring constant relation to put deflection equation in terms of force

$$\frac{F_1}{k_1} = \frac{F_2}{k_2}$$

- Substituting into equilibrium equation,

$$F - \frac{k_1}{k_2} F_2 - F_2 = 0 \quad \text{or} \quad F_2 = \frac{k_2 F}{k_1 + k_2}$$

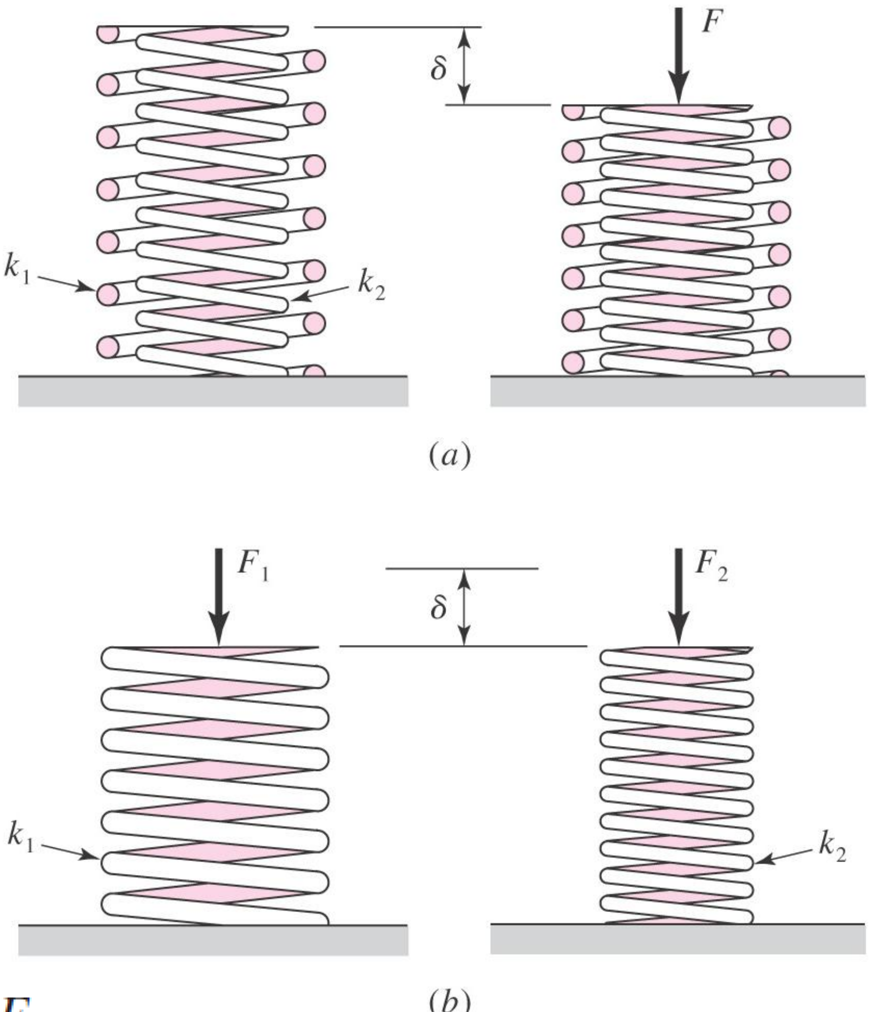


Fig. 4–15

Procedure 1 for Statically Indeterminate Problems

1. Choose the redundant reaction(s)
2. Write the equations of static equilibrium for the remaining reactions in terms of the applied loads and the redundant reaction(s).
3. Write the deflection equation(s) for the point(s) at the locations of the redundant reaction(s) in terms of the applied loads and redundant reaction(s).
4. Solve equilibrium equations and deflection equations simultaneously to determine the reactions.

Example 4-14

The indeterminate beam 11 of Appendix Table A–9 is reproduced in Fig. 4–16. Determine the reactions using procedure 1.

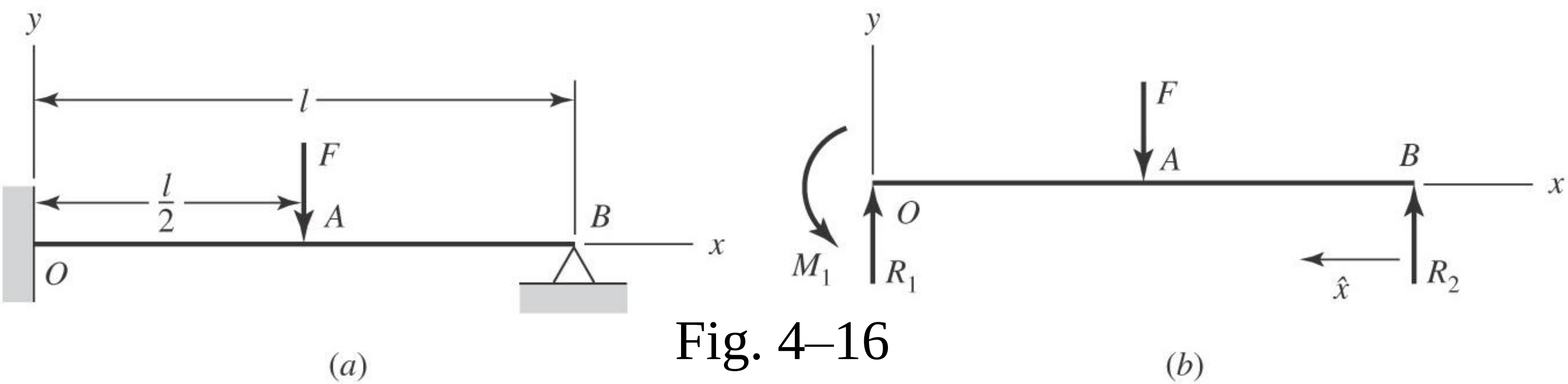


Fig. 4–16

The reactions are shown in Fig. 4–16*b*. Without R_2 the beam is a statically determinate cantilever beam. Without M_1 the beam is a statically determinate simply supported beam. In either case, the beam has only *one* redundant support. We will first solve this problem using superposition, choosing R_2 as the redundant reaction. For the second solution, we will use Castigliano's theorem with M_1 as the redundant reaction.

Example 4-14

Solution 1

1 Choose R_2 at B to be the redundant reaction.

2 Using static equilibrium equations solve for R_1 and M_1 in terms of F and R_2. This results in

$$R_1 = F - R_2 \qquad M_1 = \frac{Fl}{2} - R_2 l \tag{1}$$

3 Write the deflection equation for point B in terms of F and R_2. Using superposition of beam 1 of Table A–9 with $F = -R_2$, and beam 2 of Table A–9 with $a = l/2$, the deflection of B, at $x = l$, is

$$\delta_B = -\frac{R_2 l^2}{6EI}(l - 3l) + \frac{F(l/2)^2}{6EI}\left(\frac{l}{2} - 3l\right) = \frac{R_2 l^3}{3EI} - \frac{5Fl^3}{48EI} = 0 \tag{2}$$

4 Equation (2) can be solved for R_2 directly. This yields

$$R_2 = \frac{5F}{16} \tag{3}$$

Example 4-14

Next, substituting R_2 into Eqs. (1) completes the solution, giving

$$R_1 = \frac{11F}{16} \qquad M_1 = \frac{3Fl}{16} \tag{4}$$

Note that the solution agrees with what is given for beam 11 in Table A–9.

Example 4-14

Solution 2

1 Choose M_1 at O to be the redundant reaction.

2 Using static equilibrium equations solve for R_1 and R_2 in terms of F and M_1. This results in

$$R_1 = \frac{F}{2} + \frac{M_1}{l} \qquad R_2 = \frac{F}{2} - \frac{M_1}{l} \tag{5}$$

3 Since M_1 is the redundant reaction at O, write the equation for the angular deflection at point O. From Castigliano's theorem this is

$$\theta_O = \frac{\partial U}{\partial M_1} \tag{6}$$

Example 4-14

We can apply Eq. (4–31), using the variable x as shown in Fig. 4–16b. However, simpler terms can be found by using a variable $\hat{x}$ that starts at B and is positive to the left. With this and the expression for R_2 from Eq. (5) the moment equations are

$$M = \left(\frac{F}{2} - \frac{M_1}{l}\right)\hat{x} \qquad 0 \le \hat{x} \le \frac{l}{2} \tag{7}$$

$$M = \left(\frac{F}{2} - \frac{M_1}{l}\right)\hat{x} - F\left(\hat{x} - \frac{l}{2}\right) \qquad \frac{l}{2} \le \hat{x} \le l \tag{8}$$

For both equations

$$\frac{\partial M}{\partial M_1} = -\frac{\hat{x}}{l} \tag{9}$$

Example 4-14

Substituting Eqs. (7) to (9) in Eq. (6), using the form of Eq. (4–31) where $F_i = M_1$, gives

$$\theta_O = \frac{\partial U}{\partial M_1} = \frac{1}{EI}\left\{\int_0^{l/2}\left(\frac{F}{2}-\frac{M_1}{l}\right)\hat{x}\left(-\frac{\hat{x}}{l}\right)d\hat{x} + \int_{l/2}^{l}\left[\left(\frac{F}{2}-\frac{M_1}{l}\right)\hat{x} - F\left(\hat{x}-\frac{l}{2}\right)\right]\left(-\frac{\hat{x}}{l}\right)d\hat{x}\right\} = 0$$

Canceling $1/EIl$, and combining the first two integrals, simplifies this quite readily to

$$\left(\frac{F}{2}-\frac{M_1}{l}\right)\int_0^{l}\hat{x}^2\,d\hat{x} - F\int_{l/2}^{l}\left(\hat{x}-\frac{l}{2}\right)\hat{x}\,d\hat{x} = 0$$

Example 4-14

Integrating gives

$$\left(\frac{F}{2}-\frac{M_1}{l}\right)\frac{l^3}{3}-\frac{F}{3}\left[l^3-\left(\frac{l}{2}\right)^3\right]+\frac{Fl}{4}\left[l^2-\left(\frac{l}{2}\right)^2\right]=0$$

which reduces to

$$M_1=\frac{3Fl}{16} \tag{10}$$

4 Substituting Eq. (10) into (5) results in

$$R_1=\frac{11F}{16} \qquad R_2=\frac{5F}{16} \tag{11}$$

which again agrees with beam 11 of Table A–9.

Procedure 2 for Statically Indeterminate Problems

1. Write the equations of static equilibrium in terms of the applied loads and unknown restraint reactions.
2. Write the deflection equation in terms of the applied loads and unknown restraint reactions.
3. Apply boundary conditions to the deflection equation consistent with the restraints.
4. Solve the set of equations.

Example 4-15

The rods AD and CE shown in Fig. 4–17a each have a diameter of 10 mm. The second-area moment of beam ABC is $I = 62.5(10^3)$ mm^4. The modulus of elasticity of the material used for the rods and beam is $E = 200$ GPa. The threads at the ends of the rods are single-threaded with a pitch of 1.5 mm. The nuts are first snugly fit with bar ABC horizontal. Next the nut at A is tightened one full turn. Determine the resulting tension in each rod and the deflections of points A and C.

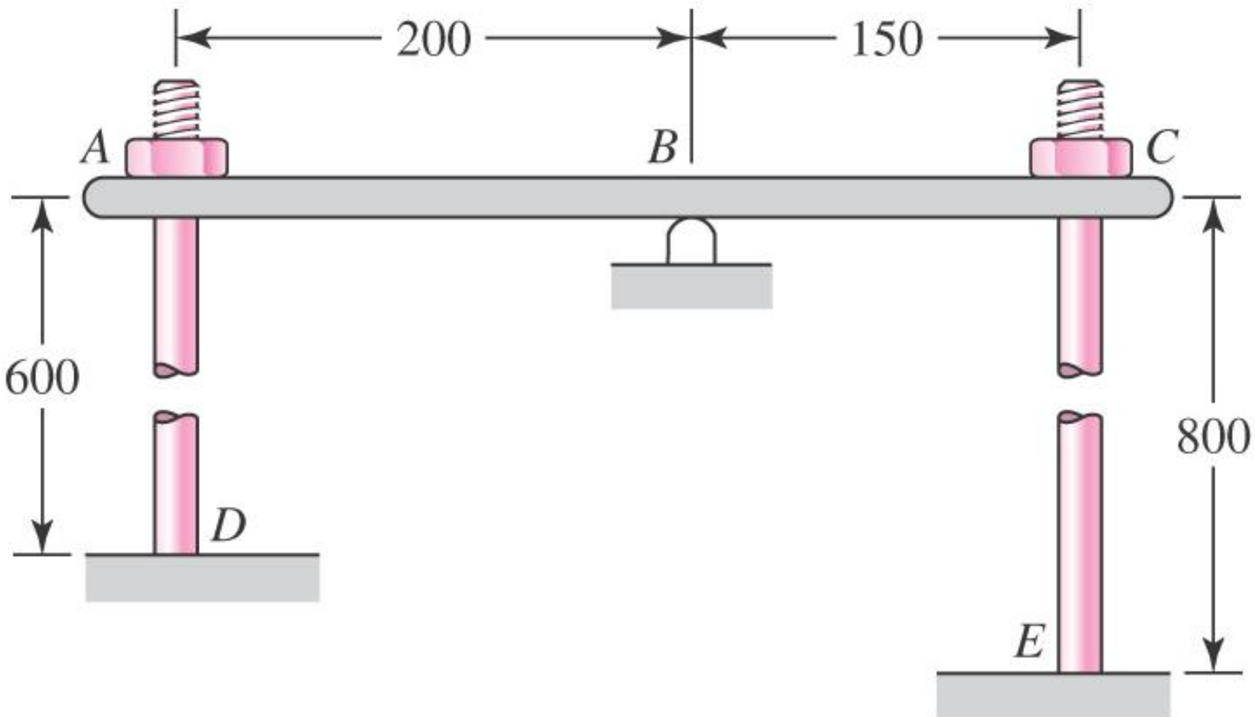


Fig. 4–17 (a)

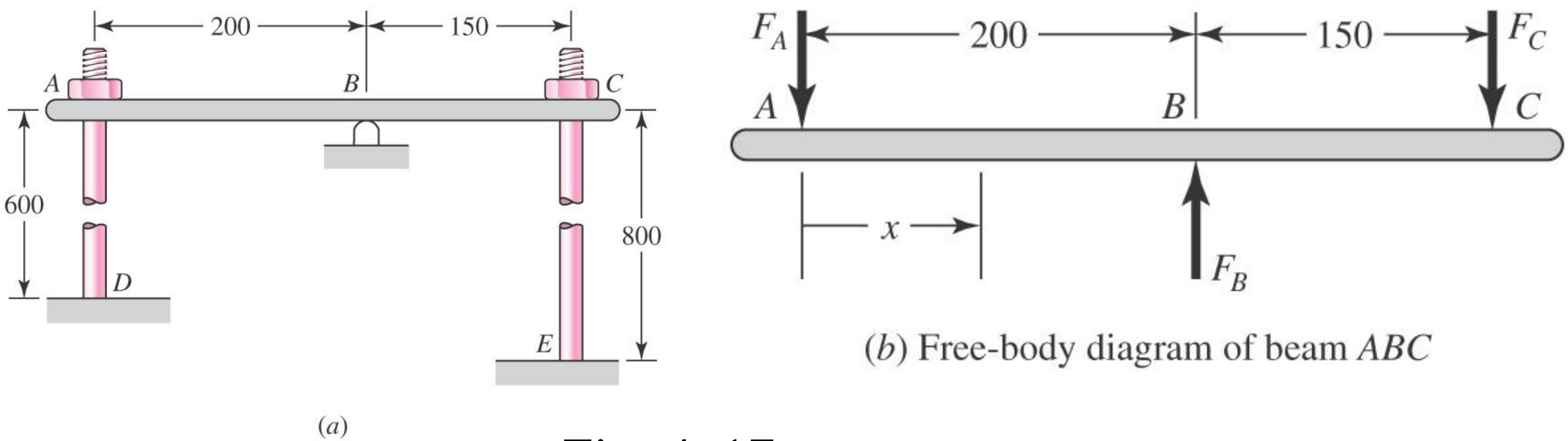
Example 4-15

There is a lot going on in this problem; a rod shortens, the rods stretch in tension, and the beam bends. Let's try the procedure!

1 The free-body diagram of the beam is shown in Fig. 4–17*b*. Summing forces, and moments about *B*, gives

$$F_B - F_A - F_C = 0 \tag{1}$$

$$4F_A - 3F_C = 0 \tag{2}$$



(*b*) Free-body diagram of beam *ABC*

Fig. 4–17

Example 4-15

2 Using singularity functions, we find the moment equation for the beam is

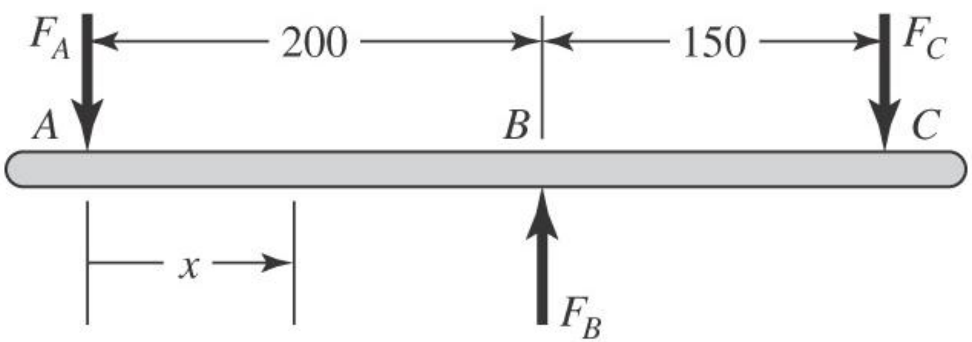
$$M = -F_A x + F_B \langle x - 0.2 \rangle^1$$

where x is in meters. Integration yields

$$EI\frac{dy}{dx} = -\frac{F_A}{2}x^2 + \frac{F_B}{2}\langle x - 0.2\rangle^2 + C_1$$

$$EIy = -\frac{F_A}{6}x^3 + \frac{F_B}{6}\langle x - 0.2\rangle^3 + C_1 x + C_2 \qquad (3)$$

The term $EI = 200(10^9)\,62.5(10^{-9}) = 1.25(10^4)\ \text{N}\cdot\text{m}^2$.



Example 4-15

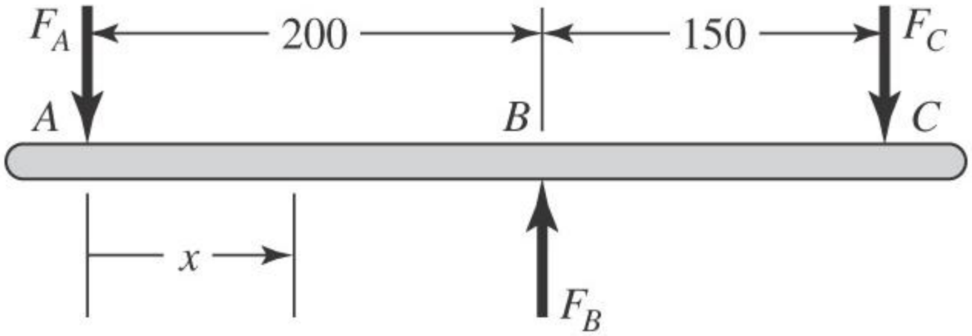
3 The upward deflection of point A is $(Fl/AE)_{AD} - Np$, where the first term is the elastic stretch of AD, N is the number of turns of the nut, and p is the pitch of the thread. Thus, the deflection of A in meters is

$$y_A = \frac{F_A(0.6)}{\frac{\pi}{4}(0.010)^2(200)(10^9)} - (1)(0.0015) \tag{4}$$

$$= 3.8197(10^{-8})F_A - 1.5(10^{-3})$$

The upward deflection of point C is $(Fl/AE)_{CE}$, or

$$y_C = \frac{F_C(0.8)}{\frac{\pi}{4}(0.010)^2(200)(10^9)} = 5.093(10^{-8})F_C \tag{5}$$



Example 4-15

Equations (4) and (5) will now serve as the boundary conditions for Eq. (3). At $x = 0$, $y = y_A$. Substituting Eq. (4) into (3) with $x = 0$ and $EI = 1.25(10^4)$, noting that the singularity function is zero for $x = 0$, gives

$$-4.7746(10^{-4})F_A + C_2 = -18.75 \tag{6}$$

At $x = 0.2$ m, $y = 0$, and Eq. (3) yields

$$-1.3333(10^{-3})F_A + 0.2C_1 + C_2 = 0 \tag{7}$$

At $x = 0.35$ m, $y = y_C$. Substituting Eq. (5) into (3) with $x = 0.35$ m and $EI = 1.25(10^4)$ gives

$$-7.1458(10^{-3})F_A + 5.625(10^{-4})F_B - 6.3662(10^{-4})F_C + 0.35C_1 + C_2 = 0 \tag{8}$$

Equations (1), (2), (6), (7), and (8) are five equations in F_A, F_B, F_C, C_1, and C_2.

Example 4-15

Written in matrix form, they are

$$\begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ 4 & 0 & -3 & 0 & 0 \\ -4.7746(10^{-4}) & 0 & 0 & 0 & 1 \\ -1.3333(10^{-3}) & 0 & 0 & 0.2 & 1 \\ -7.1458(10^{-3}) & 5.625(10^{-4}) & -6.3662(10^{-4}) & 0.35 & 1 \end{bmatrix} \begin{Bmatrix} F_A \\ F_B \\ F_C \\ C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -18.75 \\ 0 \\ 0 \end{Bmatrix}$$

Solving these equations yields

$$F_A = 2988 \text{ N} \qquad F_B = 6971 \text{ N} \qquad F_C = 3983 \text{ N}$$

$$C_1 = 106.54 \text{ N} \cdot \text{m}^2 \qquad C_2 = -17.324 \text{ N} \cdot \text{m}^3$$

Example 4-15

Equation (3) can be reduced to

$$y = -(39.84x^3 - 92.95\langle x - 0.2\rangle^3 - 8.523x + 1.386)(10^{-3})$$

$$\text{At } x = 0,\ y = y_A = -1.386(10^{-3})\ \text{m} = -1.386\ \text{mm.}$$

$$\text{At } x = 0.35\ \text{m},\ y = y_C = -[39.84(0.35)^3 - 92.95(0.35 - 0.2)^3 - 8.523(0.35)$$
$$+ 1.386](10^{-3}) = 0.203(10^{-3})\ \text{m} = 0.203\ \text{mm}$$

Compression Members

- *Column* – A member loaded in compression such that either its length or eccentric loading causes it to experience more than pure compression
- Four categories of columns
  - Long columns with central loading
  - Intermediate-length columns with central loading
  - Columns with eccentric loading
  - Struts or short columns with eccentric loading

Long Columns with Central Loading

- When *P* reaches *critical load*, column becomes *unstable* and bending develops rapidly
- Critical load depends on end conditions

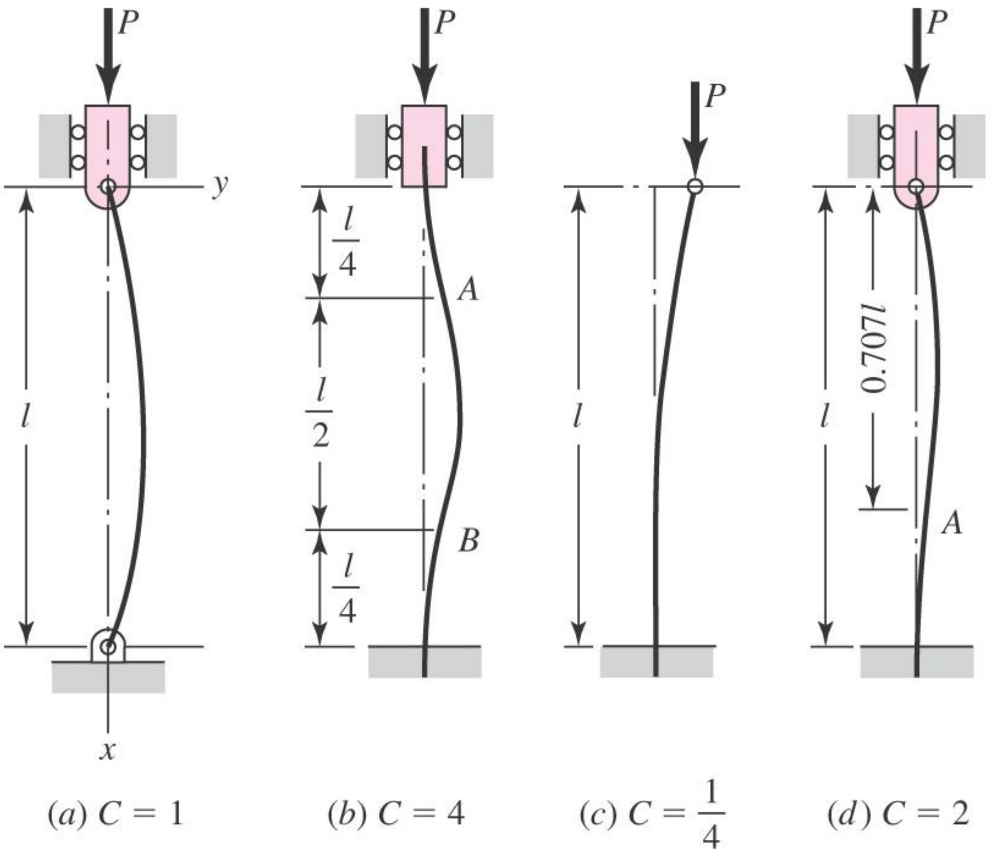


Fig. 4–18

Euler Column Formula

- For pin-ended column, critical load is given by *Euler column formula,*

$$P_{cr} = \frac{\pi^2 EI}{l^2} \qquad (4\text{-}42)$$

- Applies to other end conditions with addition of constant C for each end condition

$$P_{cr} = \frac{C\pi^2 EI}{l^2} \qquad (4\text{-}43)$$

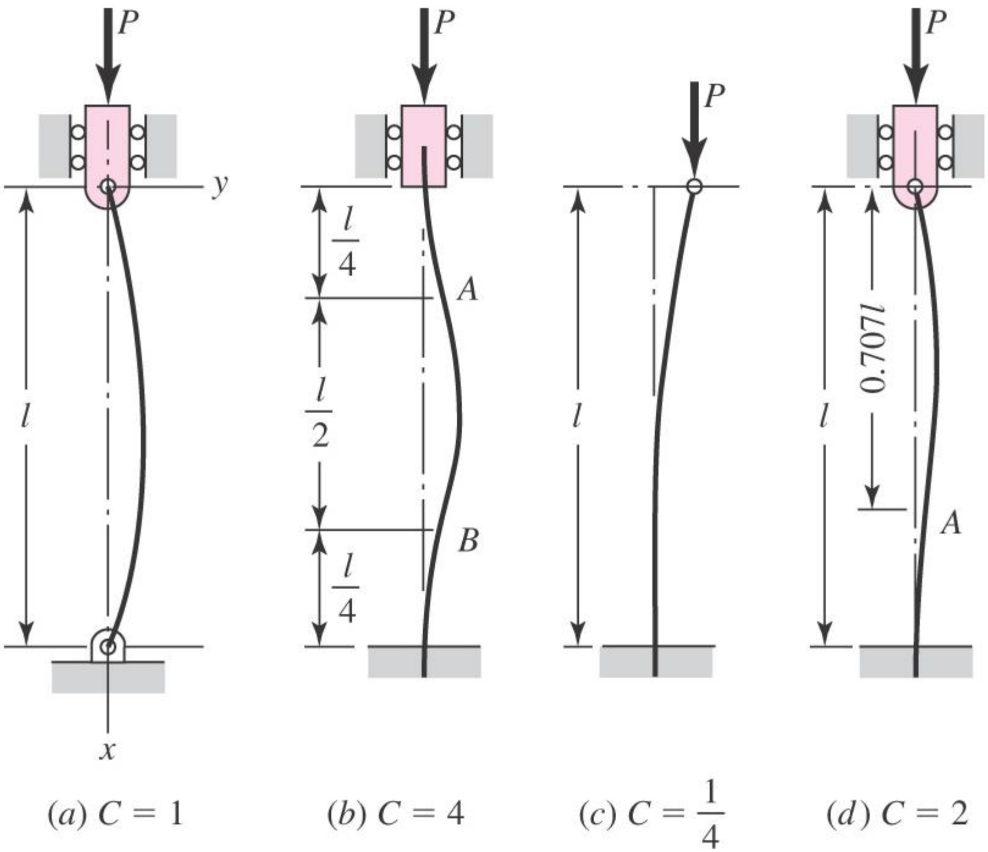
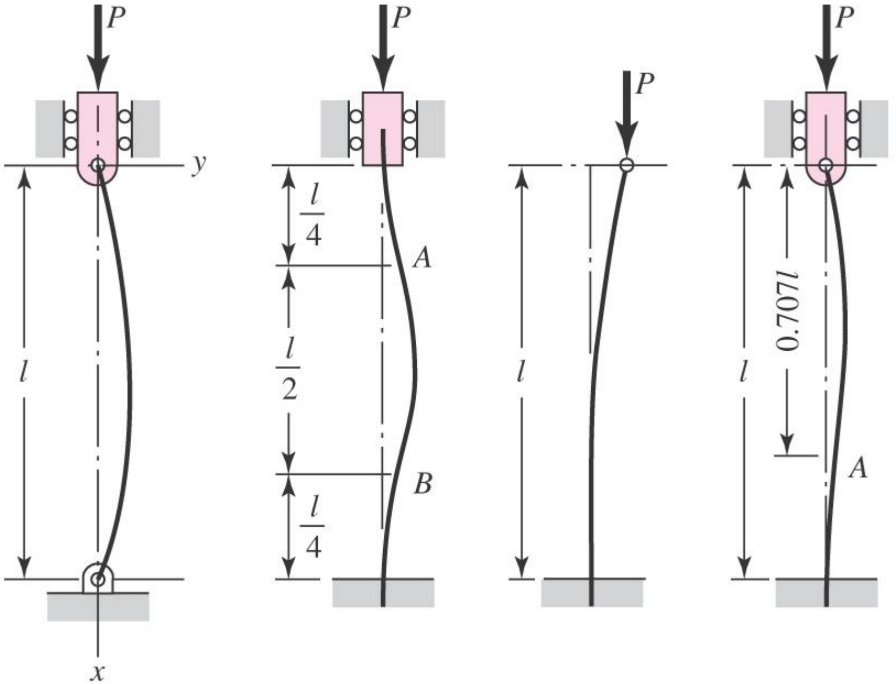


Fig. 4–18

Recommended Values for End Condition Constant

- Fixed ends are practically difficult to achieve
- More conservative values of C are often used, as in Table 4-2



(a) $C = 1$ (b) $C = 4$ (c) $C = \frac{1}{4}$ (d) $C = 2$

Table 4–2

Column End Conditions	End-Condition Constant C Theoretical Value	Conservative Value	Recommended Value*
Fixed-free	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
Rounded-rounded	1	1	1
Fixed-rounded	2	1	1.2
Fixed-fixed	4	1	1.2

*To be used only with liberal factors of safety when the column load is accurately known.

Long Columns with Central Loading

- Using $I = Ak^2$, where A is the area and k is the radius of gyration, Euler column formula can be expressed as

$$\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2} \qquad (4\text{–}44)$$

- l/k is the *slenderness ratio,* used to classify columns according to length categories.
- P_{cr}/A is the *critical unit load,* the load per unit area necessary to place the column in a condition of *unstable equilibrium.*

Euler Curve

- Plotting P_{cr}/A vs l/k, with $C = 1$ gives curve PQR

$$\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2} \tag{4–44}$$

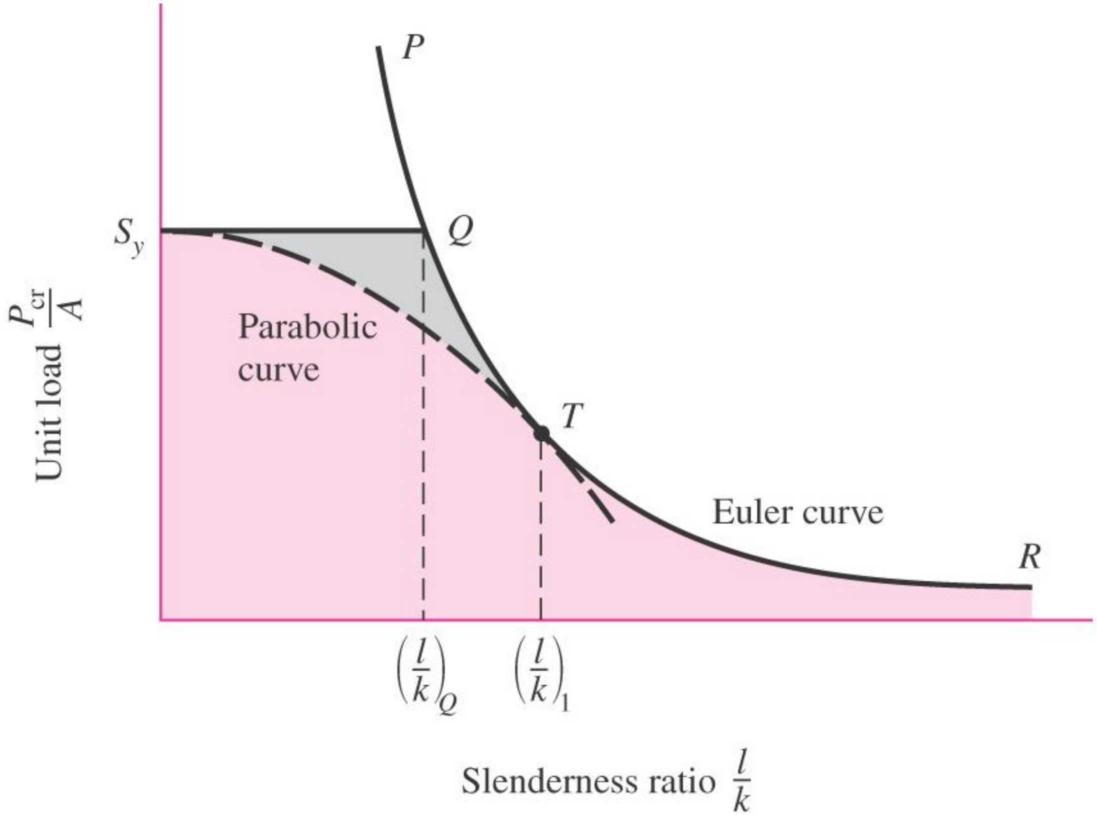


Fig. 4–19

Long Columns with Central Loading

- Tests show vulnerability to failure near point Q
- Since buckling is sudden and catastrophic, a conservative approach near Q is desired
- Point T is usually defined such that $P_{cr}/A = S_y/2$, giving

$$\left(\frac{l}{k}\right)_1 = \left(\frac{2\pi^2 C E}{S_y}\right)^{1/2} \tag{4-45}$$

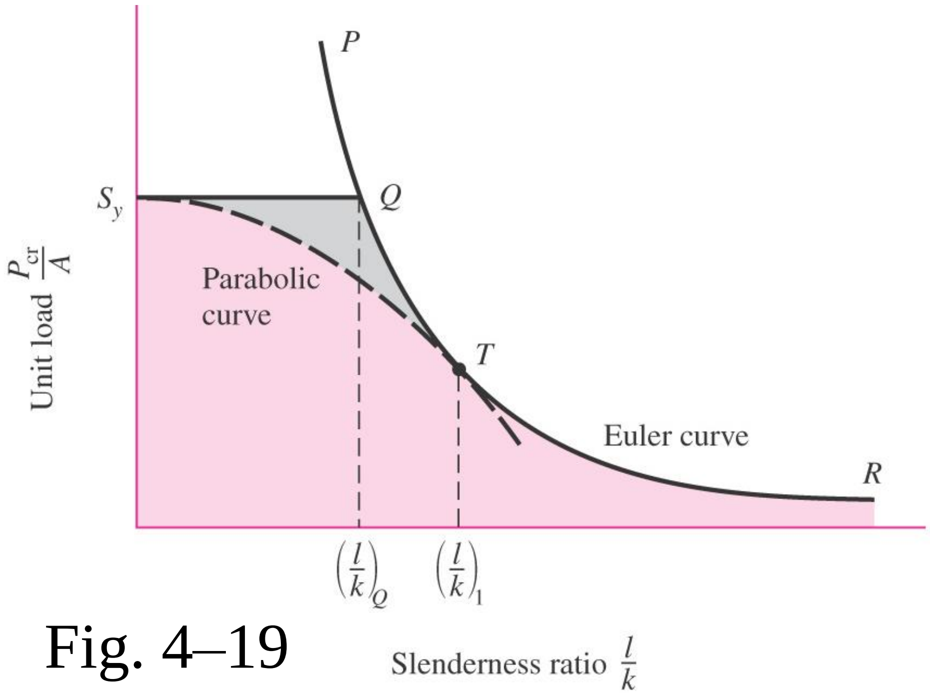


Fig. 4–19

Condition for Use of Euler Equation

- For $(l/k) > (l/k)_1$, use Euler equation
- For $(l/k) \le (l/k)_1$, use a parabolic curve between S_y and T

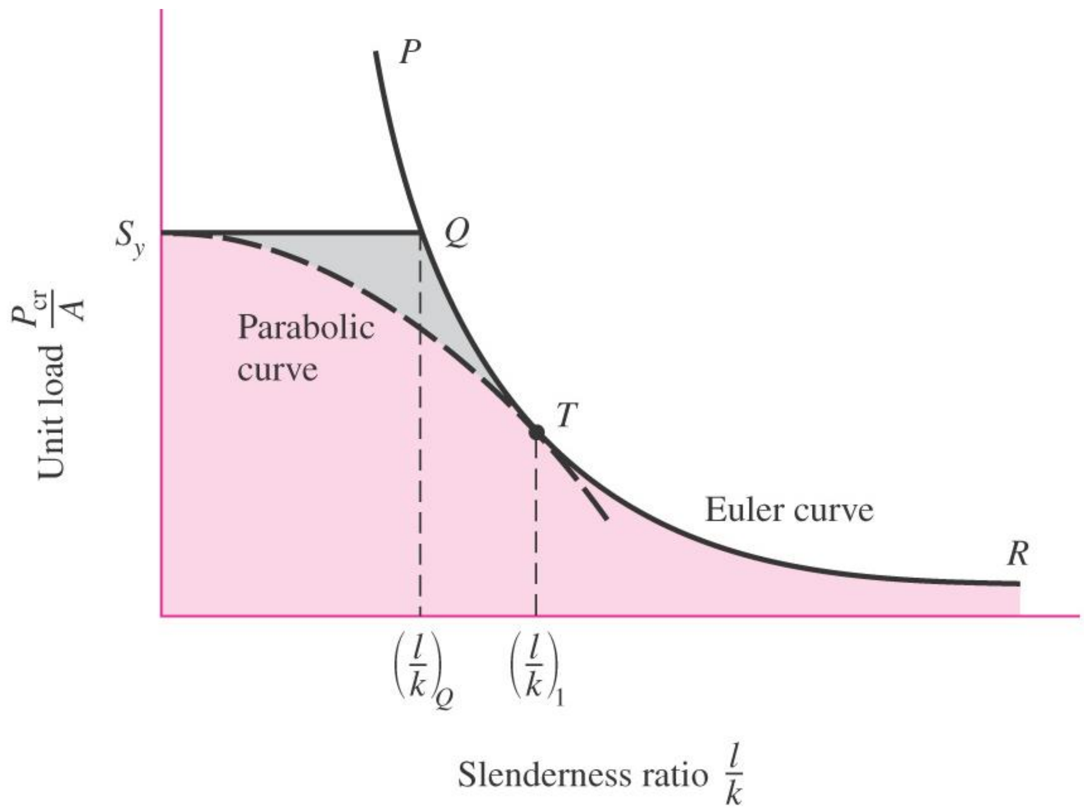


Fig. 4–19

Intermediate-Length Columns with Central Loading

- For intermediate-length columns, where $(l/k) \leq (l/k)_1$, use a parabolic curve between S_y and T
- General form of parabola

$$\frac{P_{cr}}{A} = a - b\left(\frac{l}{k}\right)^2$$

- If parabola starts at S_y, then $a = S_y$
- If parabola fits tangent to Euler curve at T, then

$$b = \left(\frac{S_y}{2\pi}\right)^2 \frac{1}{CE}$$

- Results in *parabolic formula*, also known as *J.B. Johnson formula*

$$\frac{P_{cr}}{A} = S_y - \left(\frac{S_y}{2\pi}\frac{l}{k}\right)^2 \frac{1}{CE} \qquad \frac{l}{k} \leq \left(\frac{l}{k}\right)_1 \qquad (4\text{–}46)$$

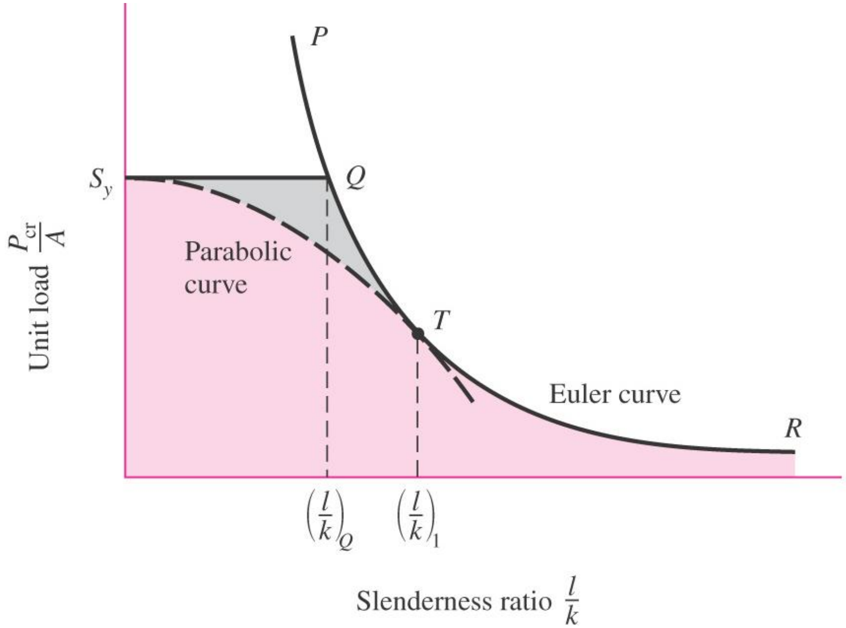


Fig. 4–19

Columns with Eccentric Loading

- For eccentrically loaded column with eccentricity *e*,

$$M = -P(e+y)$$

- Substituting into $d^2y/dx^2 = M/EI$,

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{Pe}{EI}$$

- Solving with boundary conditions $y = 0$ at $x = 0$ and at $x = l$

$$y = e\left[\tan\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)\sin\left(\sqrt{\frac{P}{EI}}x\right) + \cos\left(\sqrt{\frac{P}{EI}}x\right) - 1\right]$$

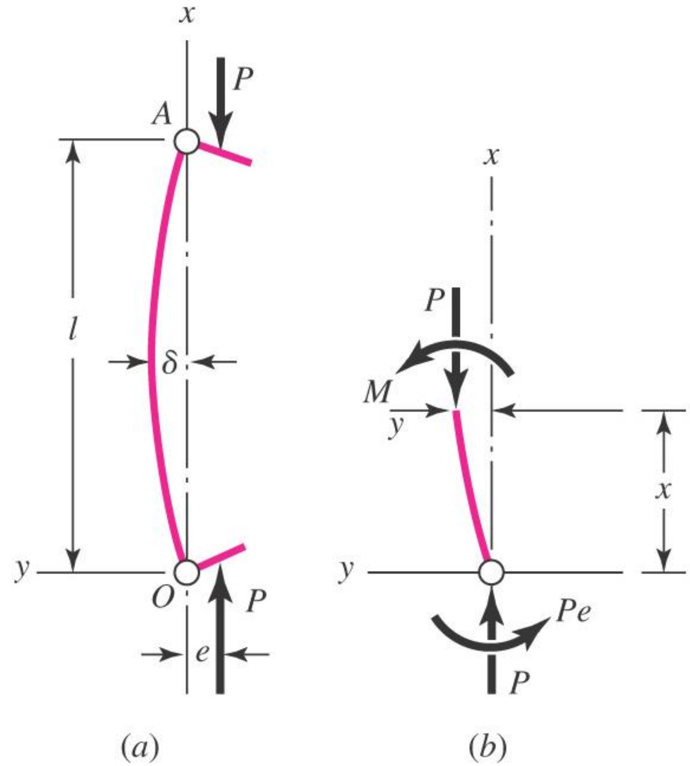


Fig. 4–20

Columns with Eccentric Loading

- At midspan where $x = l/2$

$$\delta = e\left[\sec\left(\sqrt{\frac{P}{EI}}\,\frac{l}{2}\right) - 1\right] \qquad (4\text{–}47)$$

$$M_{max} = P(e + \delta) = Pe\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) \qquad (4\text{–}48)$$

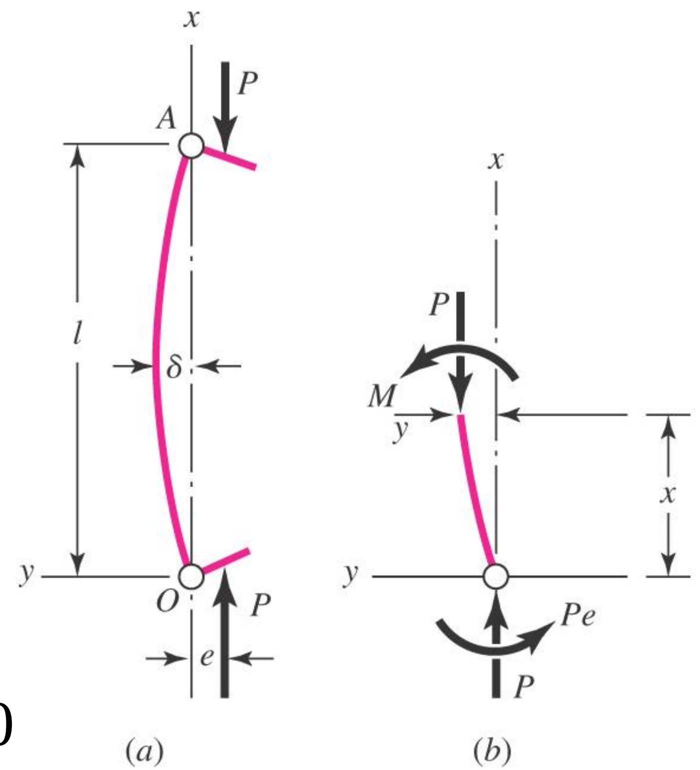
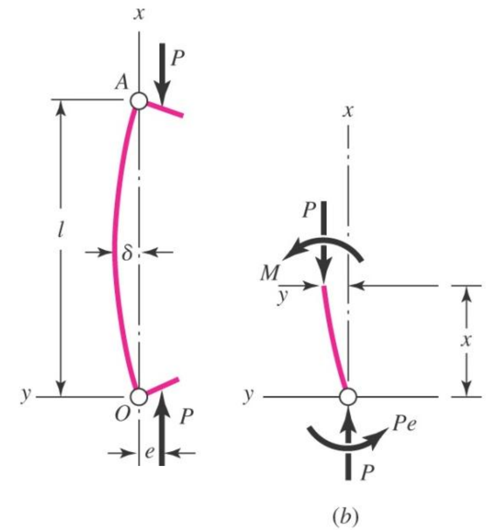


Fig. 4–20

Columns with Eccentric Loading

- The maximum compressive stress includes axial and bending

$$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Mc}{Ak^2}$$



- Substituting M_{max} from Eq. (4-48)

$$\sigma_c = \frac{P}{A}\left[1 + \frac{ec}{k^2}\sec\left(\frac{l}{2k}\sqrt{\frac{P}{EA}}\right)\right] \qquad (4\text{–}49)$$

- Using S_{yc} as the maximum value of σ_c, and solving for P/A, we obtain the *secant column formula*

$$\frac{P}{A} = \frac{S_{yc}}{1 + (ec/k^2)\sec[(l/2k)\sqrt{P/AE}]} \qquad (4\text{–}50)$$

Secant Column Formula

- Secant Column Formula

$$\frac{P}{A} = \frac{S_{yc}}{1 + (ec/k^2)\sec[(l/2k)\sqrt{P/AE}]} \qquad (4\text{–}50)$$

- ec/k^2 is the *eccentricity ratio*
- Design charts of secant column formula for various eccentricity ratio can be prepared for a given material strength

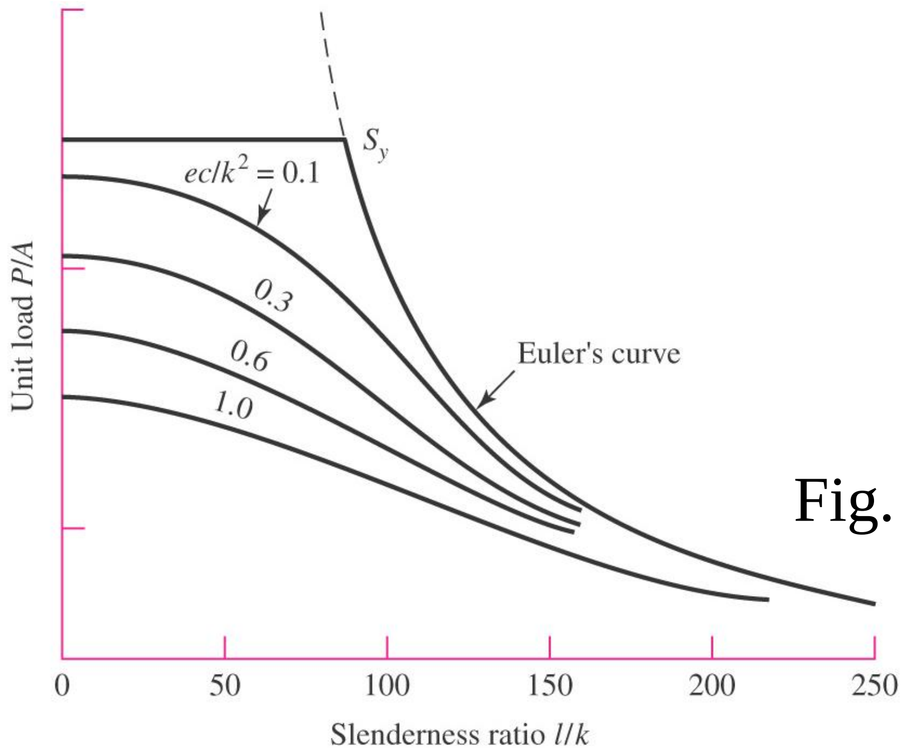


Fig. 4–21

Example 4-16

Develop specific Euler equations for the sizes of columns having
(a) Round cross sections
(b) Rectangular cross sections

Solution

(a) Using $A = \pi d^2/4$ and $k = \sqrt{I/A} = [(\pi d^4/64)/(\pi d^2/4)]^{1/2} = d/4$ with Eq. (4–44) gives

$$d = \left(\frac{64 P_{cr} l^2}{\pi^3 C E}\right)^{1/4} \qquad (4\text{–}51)$$

Example 4-16

(*b*) For the rectangular column, we specify a cross section $h \times b$ with the restriction that $h \le b$. If the end conditions are the *same* for buckling in both directions, then buckling will occur in the direction of the least thickness. Therefore

$$I = \frac{bh^3}{12} \qquad A = bh \qquad k^2 = I/A = \frac{h^2}{12}$$

Substituting these in Eq. (4–44) gives

$$b = \frac{12 P_{cr} l^2}{\pi^2 C E h^3} \qquad h \le b \tag{4–52}$$

Note, however, that rectangular columns do not generally have the same end conditions in both directions.

Example 4-17

Specify the diameter of a round column 1.5 m long that is to carry a maximum load estimated to be 22 kN. Use a design factor $n_d = 4$ and consider the ends as pinned (rounded). The column material selected has a minimum yield strength of 500 MPa and a modulus of elasticity of 207 GPa.

Solution

We shall design the column for a critical load of

$$P_{\text{cr}} = n_d P = 4(22) = 88 \text{ kN}$$

Then, using Eq. (4–51) with $C = 1$ (see Table 4–2) gives

$$d = \left(\frac{64 P_{\text{cr}} l^2}{\pi^3 C E}\right)^{1/4} = \left[\frac{64(88)(1.5)^2}{\pi^3 (1)(207)}\right]^{1/4} \left(\frac{10^3}{10^9}\right)^{1/4} (10^3) = 37.48 \text{ mm}$$

Table A–17 shows that the preferred size is 40 mm. The slenderness ratio for this size is

$$\frac{l}{k} = \frac{l}{d/4} = \frac{1.5(10^3)}{40/4} = 150$$

Example 4-17

To be sure that this is an Euler column, we use Eq. (5–51) and obtain

$$\left(\frac{l}{k}\right)_1 = \left(\frac{2\pi^2 CE}{S_y}\right)^{1/2} = \left[\frac{2\pi^2(1)(207)}{500}\right]^{1/2}\left(\frac{10^9}{10^6}\right)^{1/2} = 90.4$$

which indicates that it is indeed an Euler column. So select

$$d = 40 \text{ mm}$$

Example 4-18

Repeat Ex. 4–16 for J. B. Johnson columns.

(*a*) For round columns, Eq. (4–46) yields

$$d = 2\left(\frac{P_{\text{cr}}}{\pi S_y} + \frac{S_y l^2}{\pi^2 C E}\right)^{1/2} \tag{4–53}$$

(*b*) For a rectangular section with dimensions $h \le b$, we find

$$b = \frac{P_{\text{cr}}}{h S_y \left(1 - \dfrac{3 l^2 S_y}{\pi^2 C E h^2}\right)} \qquad h \le b \tag{4–54}$$

Example 4-19

Choose a set of dimensions for a rectangular link that is to carry a maximum compressive load of 5000 lbf. The material selected has a minimum yield strength of 75 kpsi and a modulus of elasticity $E = 30$ Mpsi. Use a design factor of 4 and an end condition constant $C = 1$ for buckling in the weakest direction, and design for (*a*) a length of 15 in, and (*b*) a length of 8 in with a minimum thickness of $\frac{1}{2}$ in.

Solution

(*a*) Using Eq. (4–44), we find the limiting slenderness ratio to be

$$\left(\frac{l}{k}\right)_1 = \left(\frac{2\pi^2 CE}{S_y}\right)^{1/2} = \left[\frac{2\pi^2(1)(30)(10^6)}{75(10)^3}\right]^{1/2} = 88.9$$

By using $P_{cr} = n_d P = 4(5000) = 20\,000$ lbf, Eqs. (4–52) and (4–54) are solved, using various values of h, to form Table 4–3. The table shows that a cross section of $\frac{5}{8}$ by $\frac{3}{4}$ in, which is marginally suitable, gives the least area.

h	b	A	l/k	Type	Eq. No.
0.375	3.46	1.298	139	Euler	(4–52)
0.500	1.46	0.730	104	Euler	(4–52)
0.625	0.76	0.475	83	Johnson	(4–54)
0.5625	1.03	0.579	92	Euler	(4–52)

Example 4-19

(*b*) An approach similar to that in part (*a*) is used with $l = 8$ in. All trial computations are found to be in the J. B. Johnson region of l/k values. A minimum area occurs when the section is a near square. Thus a cross section of $\frac{1}{2}$ by $\frac{3}{4}$ in is found to be suitable and safe.

Struts or Short Compression Members

- *Strut* - short member loaded in compression
- If eccentricity exists, maximum stress is at B with axial compression and bending.

$$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{PecA}{IA} = \frac{P}{A}\left(1 + \frac{ec}{k^2}\right) \qquad (4\text{–}55)$$

- Note that it is not a function of length
- Differs from secant equation in that it assumes small effect of bending deflection
- If bending deflection is limited to 1 percent of e, then from Eq. (4-44), the limiting slenderness ratio for strut is

$$\left(\frac{l}{k}\right)_2 = 0.282\left(\frac{AE}{P}\right)^{1/2} \qquad (4\text{–}56)$$

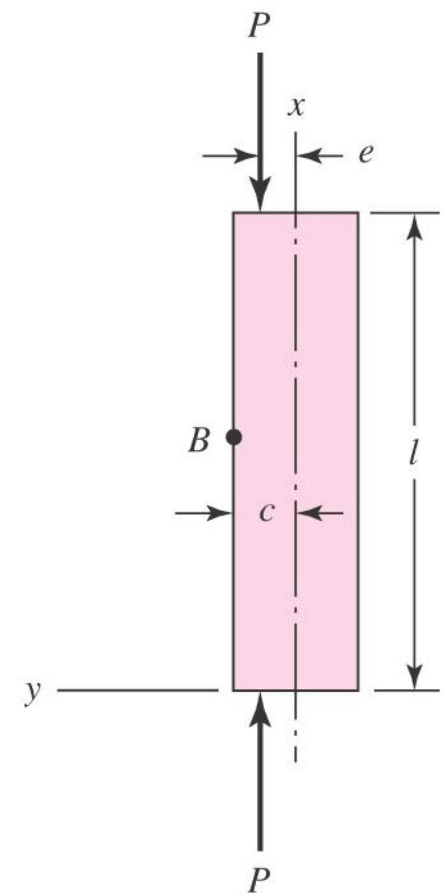


Fig. 4–22

Example 4-20

Figure 4–23*a* shows a workpiece clamped to a milling machine table by a bolt tightened to a tension of 2000 lbf. The clamp contact is offset from the centroidal axis of the strut by a distance $e = 0.10$ in, as shown in part *b* of the figure. The strut, or block, is steel, 1 in square and 4 in long, as shown. Determine the maximum compressive stress in the block.

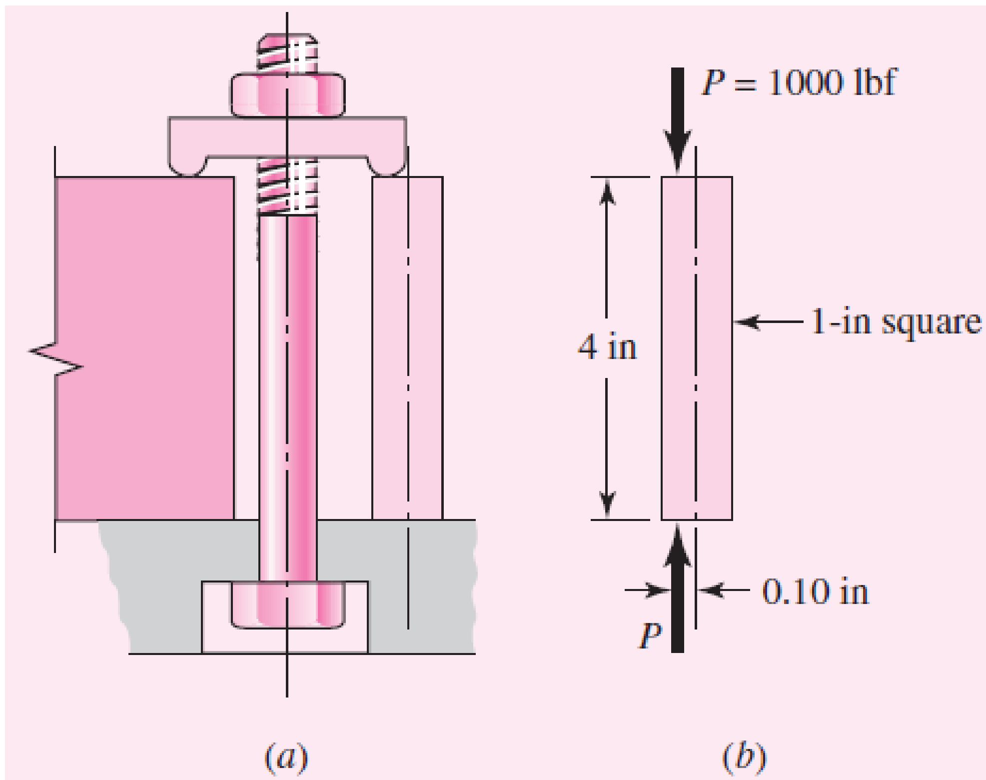


Fig. 4–23

Example 4-20

First we find $A = bh = 1(1) = 1$ in^2, $I = bh^3/12 = 1(1)^3/12 = 0.0833$ in^4, $k^2 = I/A = 0.0833/1 = 0.0833$ in^2, and $l/k = 4/(0.0833)^{1/2} = 13.9$. Equation (4–56) gives the limiting slenderness ratio as

$$\left(\frac{l}{k}\right)_2 = 0.282\left(\frac{AE}{P}\right)^{1/2} = 0.282\left[\frac{1(30)(10^6)}{1000}\right]^{1/2} = 48.8$$

Thus the block could be as long as

$$l = 48.8k = 48.8(0.0833)^{1/2} = 14.1 \text{ in}$$

before it need be treated by using the secant formula. So Eq. (4–55) applies and the maximum compressive stress is

$$\sigma_c = \frac{P}{A}\left(1 + \frac{ec}{k^2}\right) = \frac{1000}{1}\left[1 + \frac{0.1(0.5)}{0.0833}\right] = 1600 \text{ psi}$$

Elastic Stability

- Be alert for buckling instability in structural members that are
  - Loaded in compression (directly or indirectly)
  - Long or thin
  - Unbraced
- Instability may be
  - Local or global
  - Elastic or plastic
  - Bending or torsion

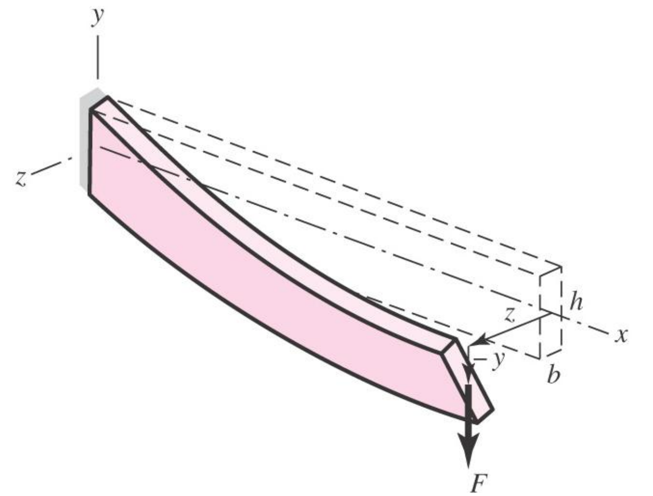


Fig. 4–24

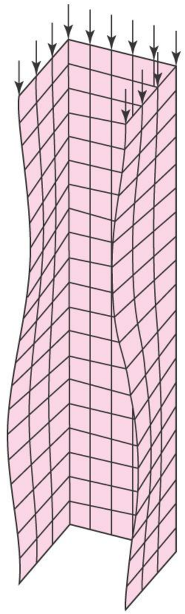


Fig. 4–25

Shock and Impact

- *Impact* – collision of two masses with initial relative velocity
- *Shock* – a suddenly applied force or disturbance

Shock and Impact

- Example of automobile collision
  - m_1 is mass of engine
  - m_2 is mass of rest of vehicle
  - Springs represent stiffnesses of various structural elements
- Equations of motion, assuming linear springs

$$m\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0$$
$$m\ddot{x}_2 + k_3 x_2 - k_2(x_1 - x_2) = 0 \qquad (4\text{–}57)$$

- Equations can be solved for any impact velocity

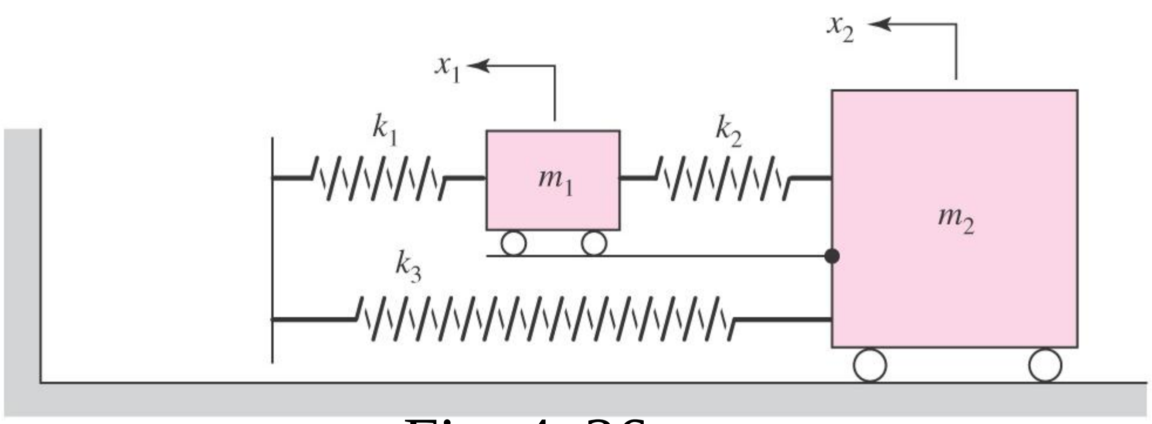


Fig. 4–26

Suddenly Applied Loading

- Weight falls distance *h* and suddenly applies a load to a cantilever beam
- Find deflection and force applied to the beam due to impact

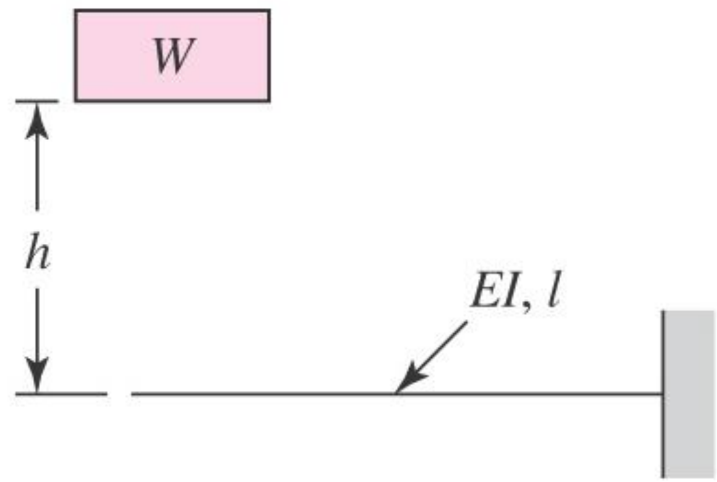


Fig. 4–27 (*a*)

Suddenly Applied Loading

- Abstract model considering beam as simple spring
- From Table A-9, beam 1,

  $k = F/y = 3EI/l^3$
- Assume the beam to be massless, so no momentum transfer, just energy transfer
- Loss of potential energy from change of elevation is

  $W(h + \delta)$
- Increase in potential energy from compressing spring is

  $k\delta^2/2$
- Conservation of energy

  $W(h + \delta) = k\delta^2/2$

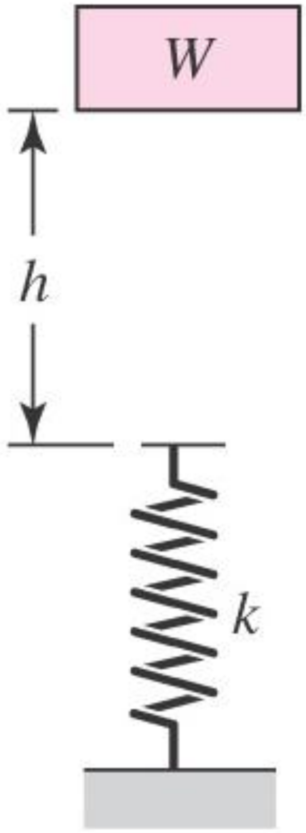


Fig. 4–27 (*b*)

Suddenly Applied Loading

- Rearranging

$$\delta^2 - 2\frac{W}{k}\delta - 2\frac{W}{k}h = 0$$

- Solving for δ

$$\delta = \frac{W}{k} \pm \frac{W}{k}\left(1 + \frac{2hk}{W}\right)^{1/2}$$

- Maximum deflection

$$\delta = \frac{W}{k} + \frac{W}{k}\left(1 + \frac{2hk}{W}\right)^{1/2} \qquad (4\text{–}58)$$

- Maximum force

$$F = k\delta = W + W\left(1 + \frac{2hk}{W}\right)^{1/2} \qquad (4\text{–}59)$$