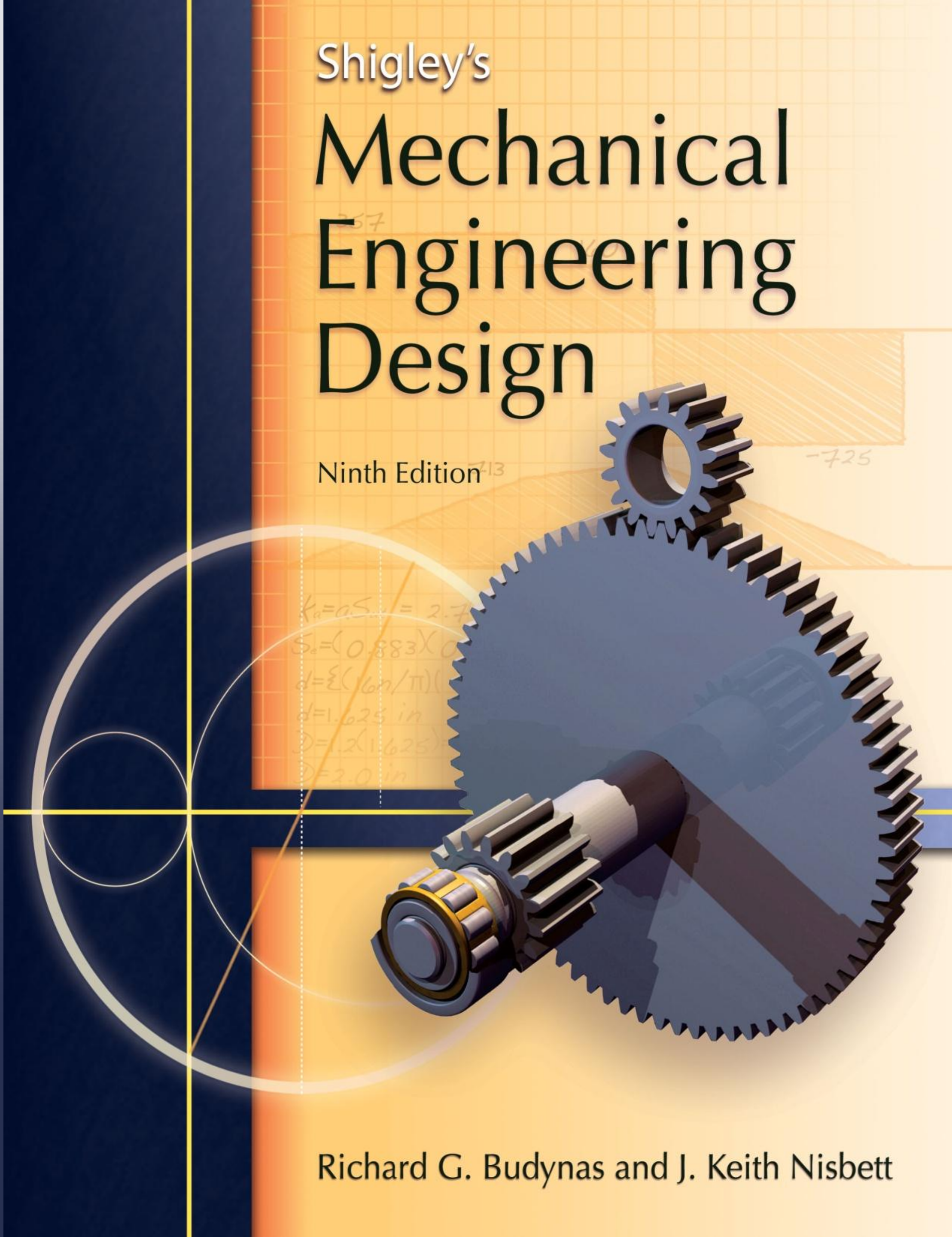
Lecture Slides

Chapter 7

Shafts and Shaft Components



Chapter Outline

Shaft Design

- Material Selection
- Geometric Layout
- Stress and strength
  - Static strength
  - Fatigue strength
- Deflection and rigidity
  - Bending deflection
  - Torsional deflection
  - Slope at bearings and shaft-supported elements
  - Shear deflection due to transverse loading of short shafts
- Vibration due to natural frequency

Shaft Materials

- Deflection primarily controlled by geometry, not material
- Stress controlled by geometry, not material
- Strength controlled by material property

Shaft Materials

- Shafts are commonly made from low carbon, CD or HR steel, such as ANSI 1020–1050 steels.
- Fatigue properties don't usually benefit much from high alloy content and heat treatment.
- Surface hardening usually only used when the shaft is being used as a bearing surface.

Shaft Materials

- Cold drawn steel typical for $d < 3$ in.
- HR steel common for larger sizes. Should be machined all over.
- Low production quantities
  - Lathe machining is typical
  - Minimum material removal may be design goal
- High production quantities
  - Forming or casting is common
  - Minimum material may be design goal

Shaft Layout

- Issues to consider for shaft layout
  - Axial layout of components
  - Supporting axial loads
  - Providing for torque transmission
  - Assembly and Disassembly

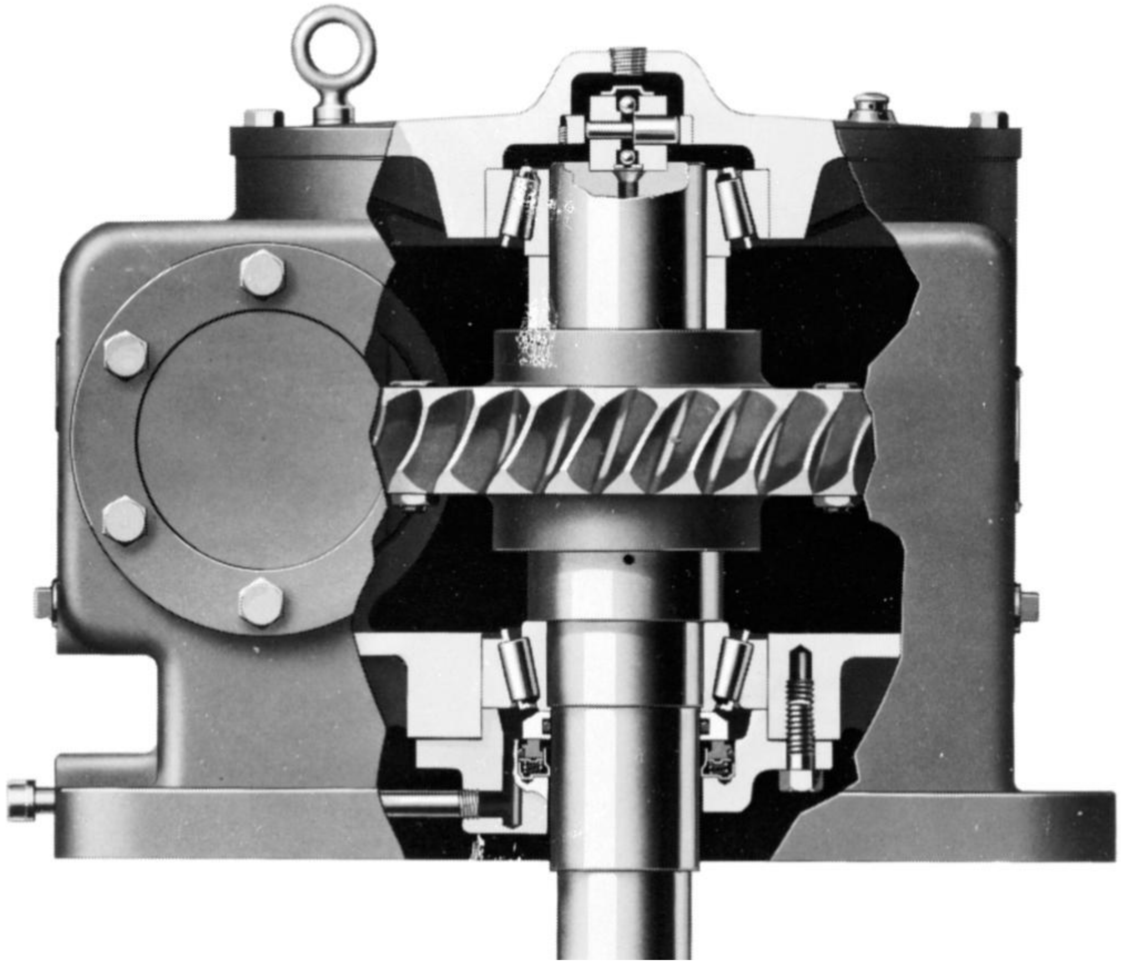


Fig. 7-1

Axial Layout of Components

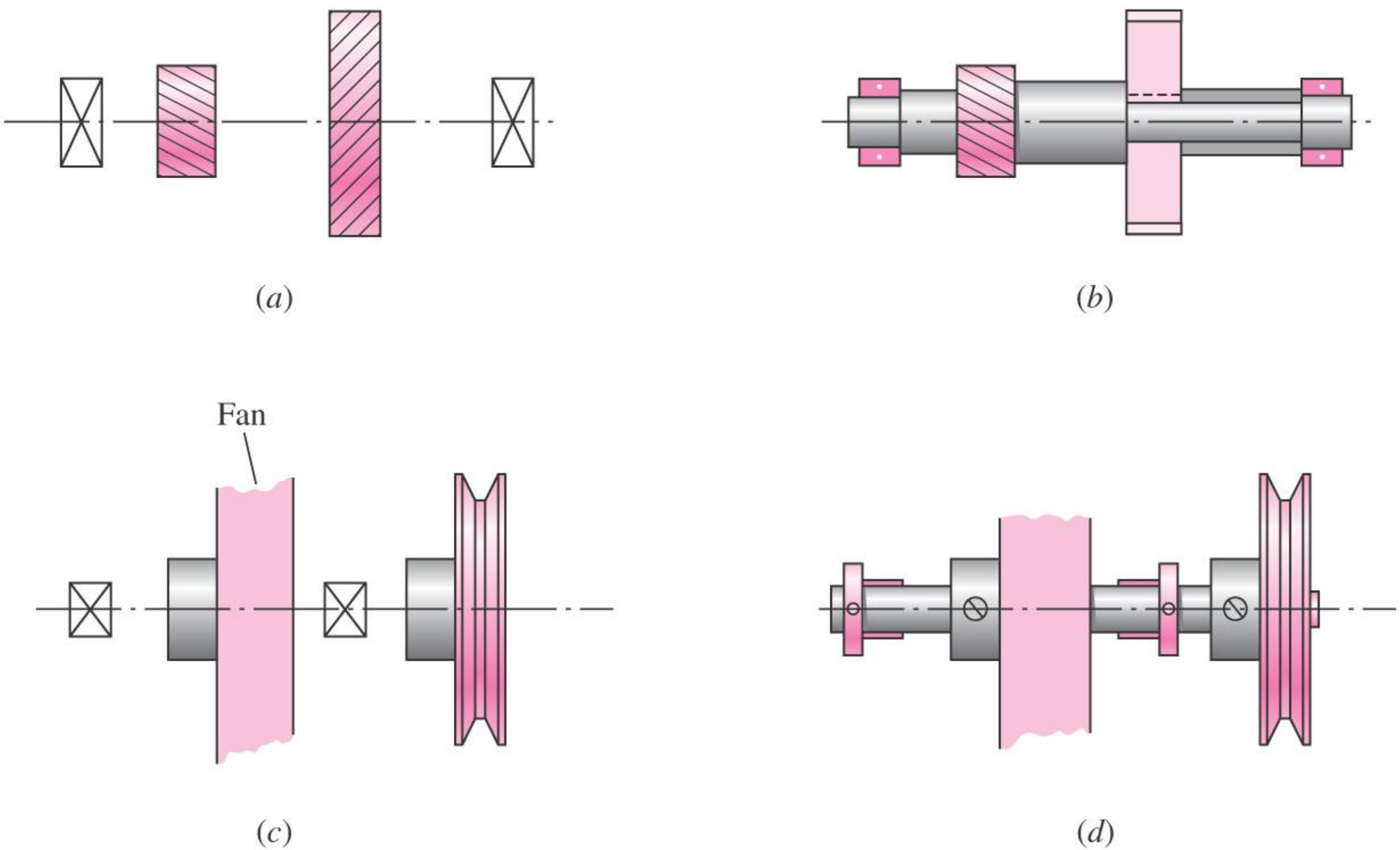


Fig. 7-2

Supporting Axial Loads

- Axial loads must be supported through a bearing to the frame.
- Generally best for only one bearing to carry axial load to shoulder
- Allows greater tolerances and prevents binding

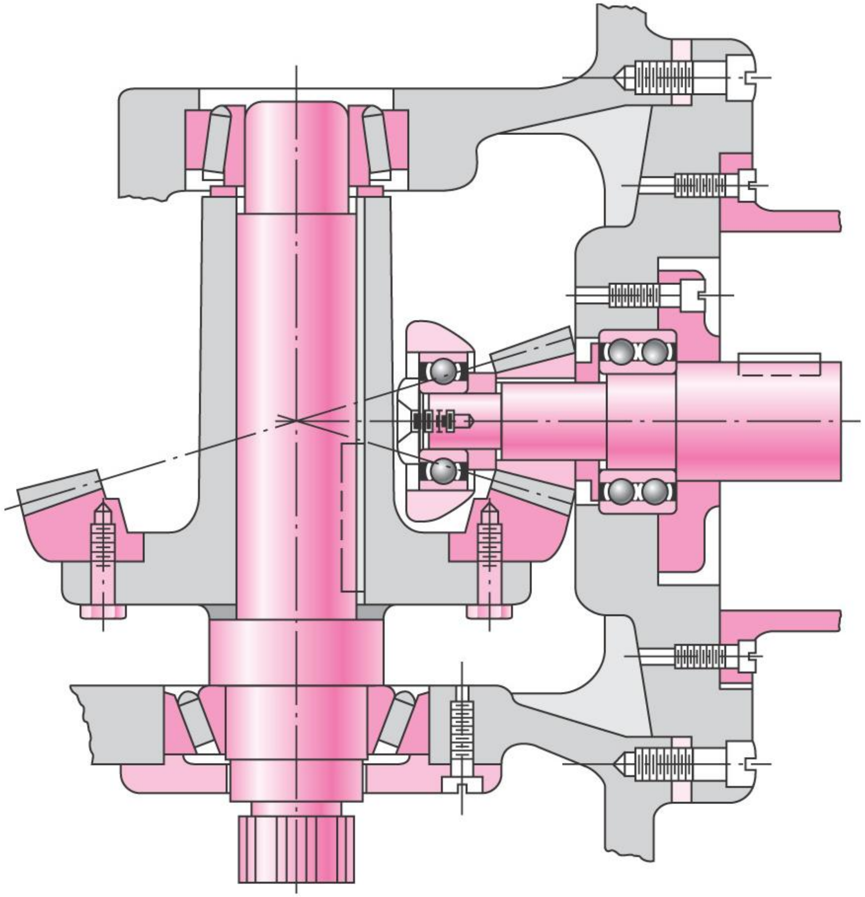


Fig. 7-4

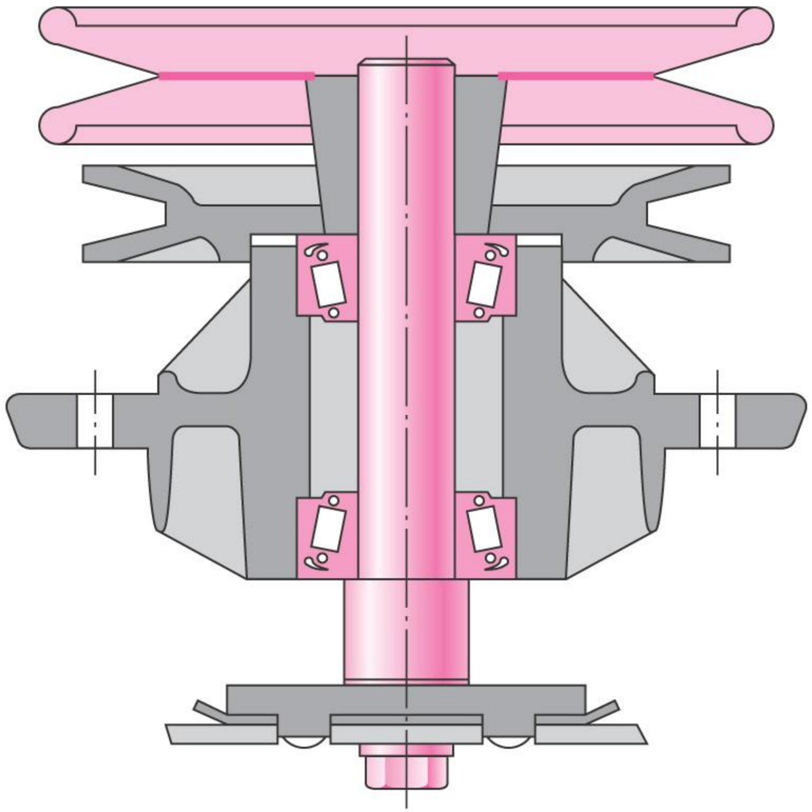


Fig. 7-3

Providing for Torque Transmission

- Common means of transferring torque to shaft
  - Keys
  - Splines
  - Setscrews
  - Pins
  - Press or shrink fits
  - Tapered fits
- Keys are one of the most effective
  - Slip fit of component onto shaft for easy assembly
  - Positive angular orientation of component
  - Can design key to be weakest link to fail in case of overload

Assembly and Disassembly

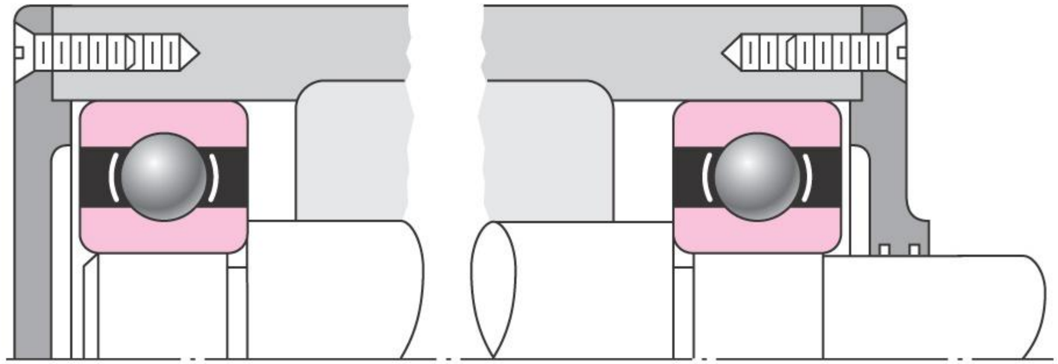


Fig. 7-5

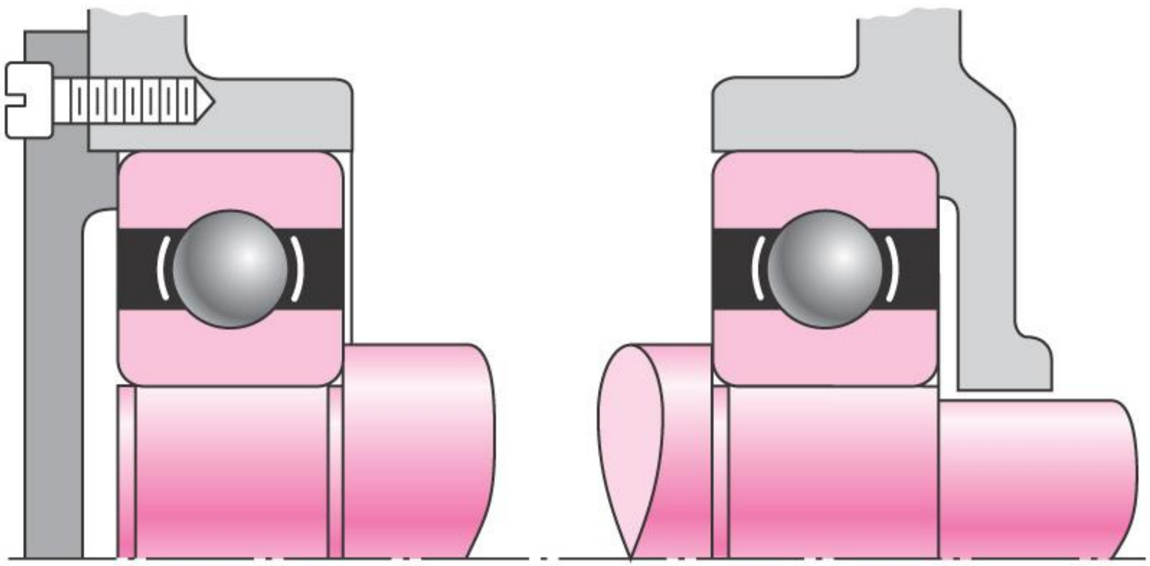


Fig. 7-6

Assembly and Disassembly

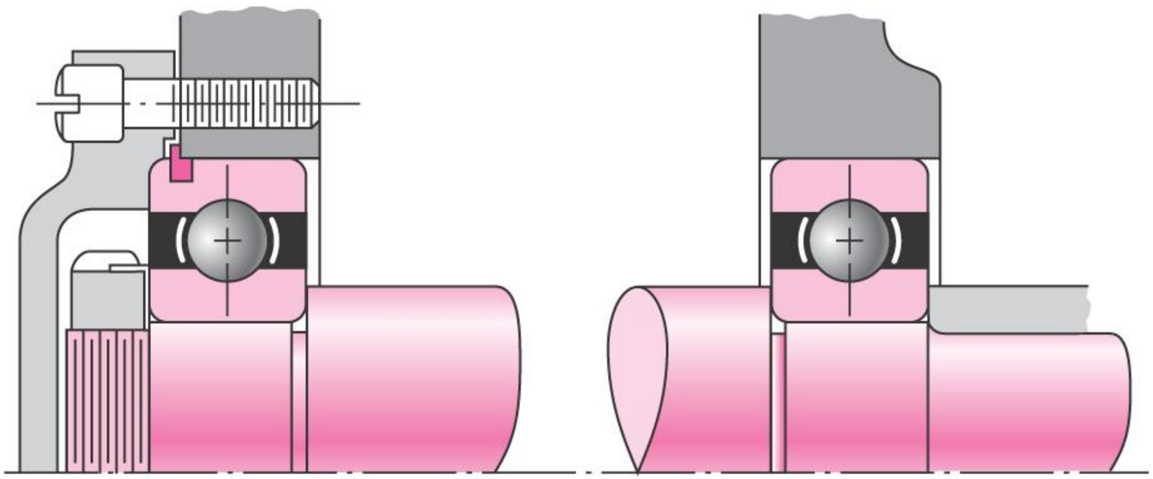


Fig. 7-7

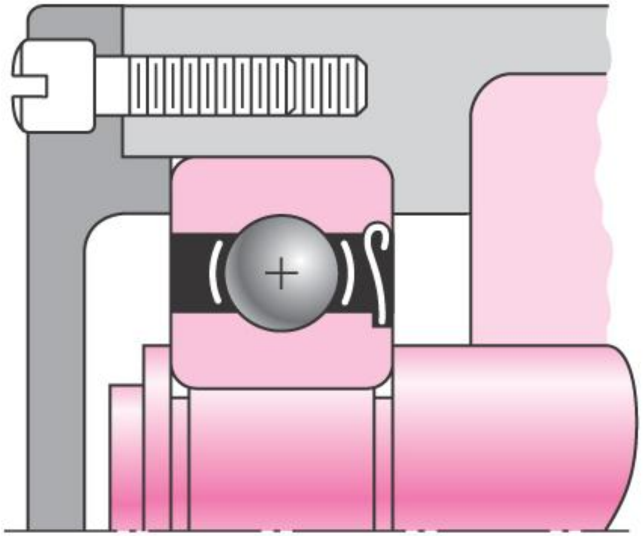


Fig. 7-8

Shaft Design for Stress

- Stresses are only evaluated at critical locations
- Critical locations are usually
  - On the outer surface
  - Where the bending moment is large
  - Where the torque is present
  - Where stress concentrations exist

Shaft Stresses

- Standard stress equations can be customized for shafts for convenience
- Axial loads are generally small and constant, so will be ignored in this section
- Standard alternating and midrange stresses

$$\sigma_a = K_f \frac{M_a c}{I} \qquad \sigma_m = K_f \frac{M_m c}{I} \qquad (7\text{–}1)$$

$$\tau_a = K_{fs} \frac{T_a c}{J} \qquad \tau_m = K_{fs} \frac{T_m c}{J} \qquad (7\text{–}2)$$

- Customized for round shafts

$$\sigma_a = K_f \frac{32 M_a}{\pi d^3} \qquad \sigma_m = K_f \frac{32 M_m}{\pi d^3} \qquad (7\text{–}3)$$

$$\tau_a = K_{fs} \frac{16 T_a}{\pi d^3} \qquad \tau_m = K_{fs} \frac{16 T_m}{\pi d^3} \qquad (7\text{–}4)$$

Shaft Stresses

- Combine stresses into von Mises stresses

$$\sigma'_a = (\sigma_a^2 + 3\tau_a^2)^{1/2} = \left[\left(\frac{32K_f M_a}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs} T_a}{\pi d^3}\right)^2\right]^{1/2} \tag{7-5}$$

$$\sigma'_m = (\sigma_m^2 + 3\tau_m^2)^{1/2} = \left[\left(\frac{32K_f M_m}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs} T_m}{\pi d^3}\right)^2\right]^{1/2} \tag{7-6}$$

Shaft Stresses

- Substitute von Mises stresses into failure criteria equation. For example, using modified Goodman line,

$$\frac{1}{n} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}}$$

$$\frac{1}{n} = \frac{16}{\pi d^3}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{ut}}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\} \quad (7\text{–}7)$$

- Solving for d is convenient for design purposes

$$d = \left(\frac{16n}{\pi}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{ut}}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\}\right)^{1/3} \quad (7\text{–}8)$$

Shaft Stresses

- Similar approach can be taken with any of the fatigue failure criteria
- Equations are referred to by referencing both the Distortion Energy method of combining stresses and the fatigue failure locus name. For example, *DE-Goodman*, *DE-Gerber,* etc.
- In analysis situation, can either use these customized equations for factor of safety, or can use standard approach from Ch. 6.
- In design situation, customized equations for *d* are much more convenient.

Shaft Stresses

- *DE-Gerber*

$$\frac{1}{n} = \frac{8A}{\pi d^3 S_e}\left\{1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}}\right)^2\right]^{1/2}\right\} \quad (7\text{–}9)$$

$$d = \left(\frac{8nA}{\pi S_e}\left\{1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}}\right)^2\right]^{1/2}\right\}\right)^{1/3} \quad (7\text{–}10)$$

where

$$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$$

$$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2}$$

Shaft Stresses

- *DE-ASME Elliptic*

$$\frac{1}{n} = \frac{16}{\pi d^3}\left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_a}{S_e}\right)^2 + 4\left(\frac{K_f M_m}{S_y}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2\right]^{1/2}$$

(7–11)

$$d = \left\{\frac{16n}{\pi}\left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_a}{S_e}\right)^2 + 4\left(\frac{K_f M_m}{S_y}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2\right]^{1/2}\right\}^{1/3}$$

(7–12)

- *DE-Soderberg*

$$\frac{1}{n} = \frac{16}{\pi d^3}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{yt}}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\}$$

(7–13)

$$d = \left(\frac{16n}{\pi}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{yt}}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\}\right)^{1/3}$$

(7–14)

Shaft Stresses for Rotating Shaft

- For rotating shaft with steady bending and torsion
  - Bending stress is completely reversed, since a stress element on the surface cycles from equal tension to compression during each rotation
  - Torsional stress is steady
  - Previous equations simplify with M_m and T_a equal to 0

Checking for Yielding in Shafts

- Always necessary to consider static failure, even in fatigue situation
- Soderberg criteria inherently guards against yielding
- ASME-Elliptic criteria takes yielding into account, but is not entirely conservative
- Gerber and modified Goodman criteria require specific check for yielding

Checking for Yielding in Shafts

- Use von Mises maximum stress to check for yielding,

$$\sigma'_{\max} = \left[(\sigma_m + \sigma_a)^2 + 3\,(\tau_m + \tau_a)^2\right]^{1/2}$$

$$= \left[\left(\frac{32K_f\,(M_m + M_a)}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs}\,(T_m + T_a)}{\pi d^3}\right)^2\right]^{1/2}$$

(7–15)

$$n_y = \frac{S_y}{\sigma'_{\max}} \qquad (7\text{–}16)$$

- Alternate simple check is to obtain conservative estimate of $\sigma'_{\max}$ by summing σ'_a and σ'_m

$$\sigma'_{\max} \approx \sigma'_a + \sigma'_m$$

Example 7-1

At a machined shaft shoulder the small diameter d is 1.100 in, the large diameter D is 1.65 in, and the fillet radius is 0.11 in. The bending moment is 1260 lbf · in and the steady torsion moment is 1100 lbf · in. The heat-treated steel shaft has an ultimate strength of $S_{ut} = 105$ kpsi and a yield strength of $S_y = 82$ kpsi. The reliability goal is 0.99.

(a) Determine the fatigue factor of safety of the design using each of the fatigue failure criteria described in this section.

(b) Determine the yielding factor of safety.

Example 7-1

(*a*) $D/d = 1.65/1.100 = 1.50$, $r/d = 0.11/1.100 = 0.10$, $K_t = 1.68$ (Fig. A–15–9), $K_{ts} = 1.42$ (Fig. A–15–8), $q = 0.85$ (Fig. 6–20), $q_{\text{shear}} = 0.88$ (Fig. 6–21).

From Eq. (6–32),

$$K_f = 1 + 0.85(1.68 - 1) = 1.58$$

$$K_{fs} = 1 + 0.88(1.42 - 1) = 1.37$$

Eq. (6–8): $$S'_e = 0.5(105) = 52.5\,\text{kpsi}$$

Eq. (6–19): $$k_a = 2.70(105)^{-0.265} = 0.787$$

Eq. (6–20): $$k_b = \left(\frac{1.100}{0.30}\right)^{-0.107} = 0.870$$

$$k_c = k_d = k_f = 1$$

Table 6–6: $$k_e = 0.814$$

$$S_e = 0.787(0.870)0.814(52.5) = 29.3\,\text{kpsi}$$

Example 7-1

For a rotating shaft, the constant bending moment will create a completely reversed bending stress.

$$M_a = 1260 \text{ lbf} \cdot \text{in} \qquad T_m = 1100 \text{ lbf} \cdot \text{in} \qquad M_m = T_a = 0$$

Applying Eq. (7–7) for the DE-Goodman criteria gives

$$\frac{1}{n} = \frac{16}{\pi(1.1)^3}\left\{\frac{\left[4\,(1.58 \cdot 1260)^2\right]^{1/2}}{29\,300} + \frac{\left[3\,(1.37 \cdot 1100)^2\right]^{1/2}}{105\,000}\right\} = 0.615$$

$$n = 1.63 \qquad \text{DE-Goodman}$$

Similarly, applying Eqs. (7–9), (7–11), and (7–13) for the other failure criteria,

$$n = 1.87 \qquad \text{DE-Gerber}$$

$$n = 1.88 \qquad \text{DE-ASME Elliptic}$$

$$n = 1.56 \qquad \text{DE-Soderberg}$$

Example 7-1

For comparison, consider an equivalent approach of calculating the stresses and applying the fatigue failure criteria directly. From Eqs. (7–5) and (7–6),

$$\sigma_a' = \left[\left(\frac{32 \cdot 1.58 \cdot 1260}{\pi\,(1.1)^3}\right)^2\right]^{1/2} = 15\,235 \text{ psi}$$

$$\sigma_m' = \left[3\left(\frac{16 \cdot 1.37 \cdot 1100}{\pi\,(1.1)^3}\right)^2\right]^{1/2} = 9988 \text{ psi}$$

Taking, for example, the Goodman failure critera, application of Eq. (6–46) gives

$$\frac{1}{n} = \frac{\sigma_a'}{S_e} + \frac{\sigma_m'}{S_{ut}} = \frac{15\,235}{29\,300} + \frac{9988}{105\,000} = 0.615$$

$$n = 1.63$$

which is identical with the previous result. The same process could be used for the other failure criteria.

Example 7-1

(*b*) For the yielding factor of safety, determine an equivalent von Mises maximum stress using Eq. (7–15).

$$\sigma'_{\max} = \left[\left(\frac{32(1.58)\,(1260)}{\pi\,(1.1)^3}\right)^2 + 3\left(\frac{16(1.37)\,(1100)}{\pi\,(1.1)^3}\right)^2\right]^{1/2} = 18\ 220\,\text{psi}$$

$$n_y = \frac{S_y}{\sigma'_{\max}} = \frac{82\,000}{18\,220} = 4.50$$

For comparison, a quick and very conservative check on yielding can be obtained by replacing $\sigma'_{\max}$ with $\sigma'_a + \sigma'_m$. This just saves the extra time of calculating $\sigma'_{\max}$ if σ'_a and σ'_m have already been determined. For this example,

$$n_y = \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{82\,000}{15\,235 + 9988} = 3.25$$

which is quite conservative compared with $n_y = 4.50$.

Estimating Stress Concentrations

- Stress analysis for shafts is highly dependent on stress concentrations.
- Stress concentrations depend on size specifications, which are not known the first time through a design process.
- Standard shaft elements such as shoulders and keys have standard proportions, making it possible to estimate stress concentrations factors before determining actual sizes.

Estimating Stress Concentrations

Table 7–1

First Iteration Estimates for Stress-Concentration Factors K_t and K_{ts}.

Warning: These factors are only estimates for use when actual dimensions are not yet determined. Do *not* use these once actual dimensions are available.

	Bending	Torsional	Axial
Shoulder fillet—sharp ($r/d = 0.02$)	2.7	2.2	3.0
Shoulder fillet—well rounded ($r/d = 0.1$)	1.7	1.5	1.9
End-mill keyseat ($r/d = 0.02$)	2.14	3.0	—
Sled runner keyseat	1.7	—	—
Retaining ring groove	5.0	3.0	5.0

Missing values in the table are not readily available.

Reducing Stress Concentration at Shoulder Fillet

- Bearings often require relatively sharp fillet radius at shoulder
- If such a shoulder is the location of the critical stress, some manufacturing techniques are available to reduce the stress concentration
  - (a) Large radius undercut into shoulder
  - (b) Large radius relief groove into back of shoulder
  - (c) Large radius relief groove into small diameter of shaft

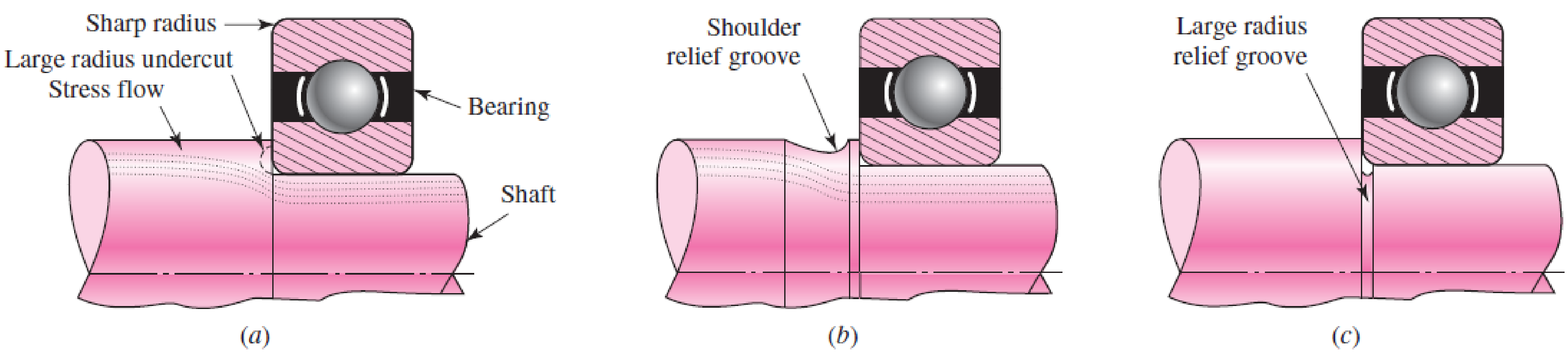


Fig. 7-9

Example 7-2

This example problem is part of a larger case study. See Chap. 18 for the full context.

A double reduction gearbox design has developed to the point that the general layout and axial dimensions of the countershaft carrying two spur gears has been proposed, as shown in Fig. 7–10. The gears and bearings are located and supported by shoulders, and held in place by retaining rings. The gears transmit torque through keys. Gears have been specified as shown, allowing the tangential and radial forces transmitted through the gears to the shaft to be determined as follows.

$$W_{23}^t = 540\,\text{lbf} \qquad W_{54}^t = 2431\,\text{lbf}$$

$$W_{23}^r = 197\,\text{lbf} \qquad W_{54}^r = 885\,\text{lbf}$$

where the superscripts *t* and *r* represent tangential and radial directions, respectively; and, the subscripts 23 and 54 represent the forces exerted *by* gears 2 and 5 (not shown) *on* gears 3 and 4, respectively.

Proceed with the next phase of the design, in which a suitable material is selected, and appropriate diameters for each section of the shaft are estimated, based on providing sufficient fatigue and static stress capacity for infinite life of the shaft, with minimum safety factors of 1.5.

Example 7-2

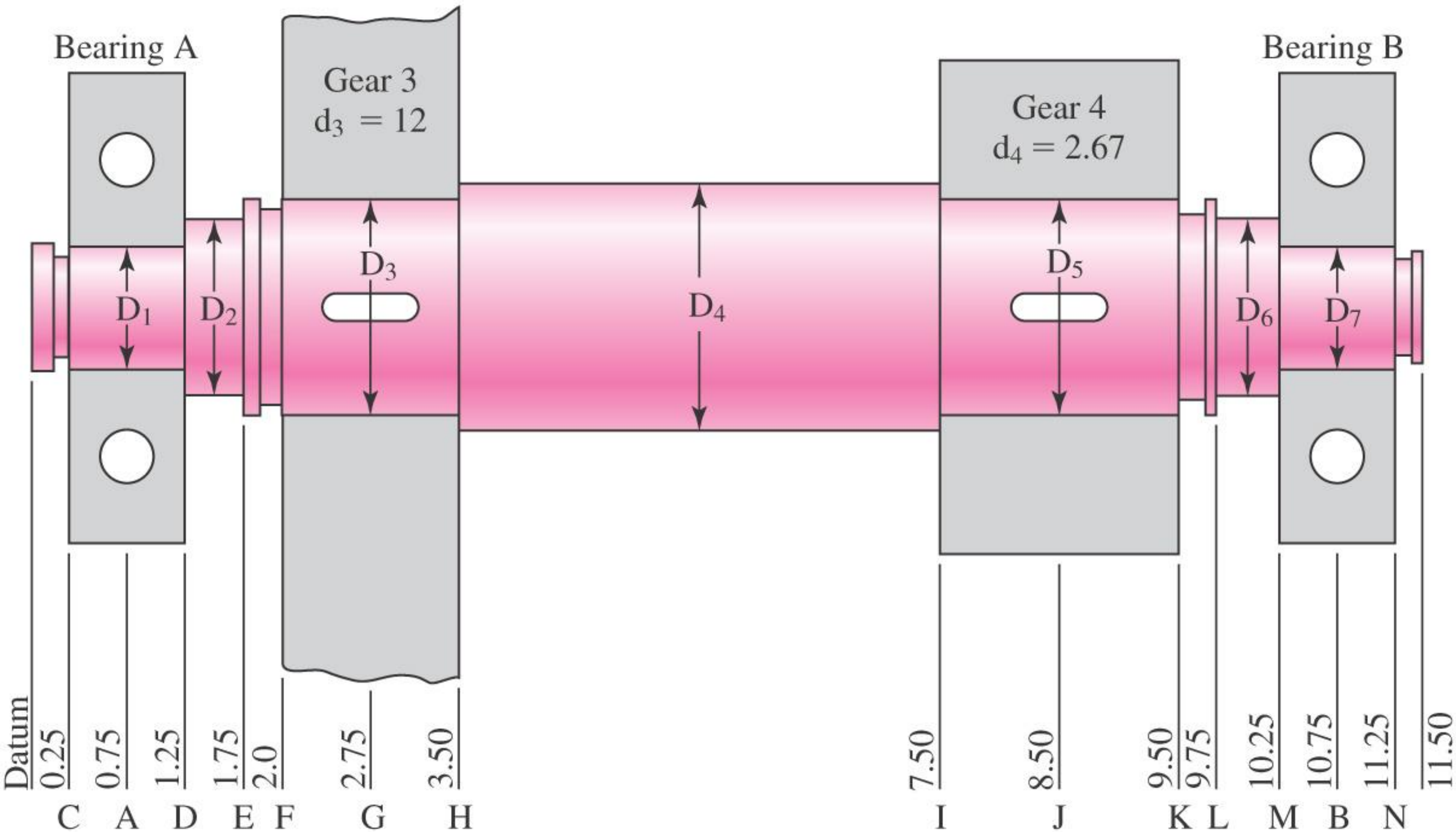


Fig. 7-10

Example 7-2

Solution

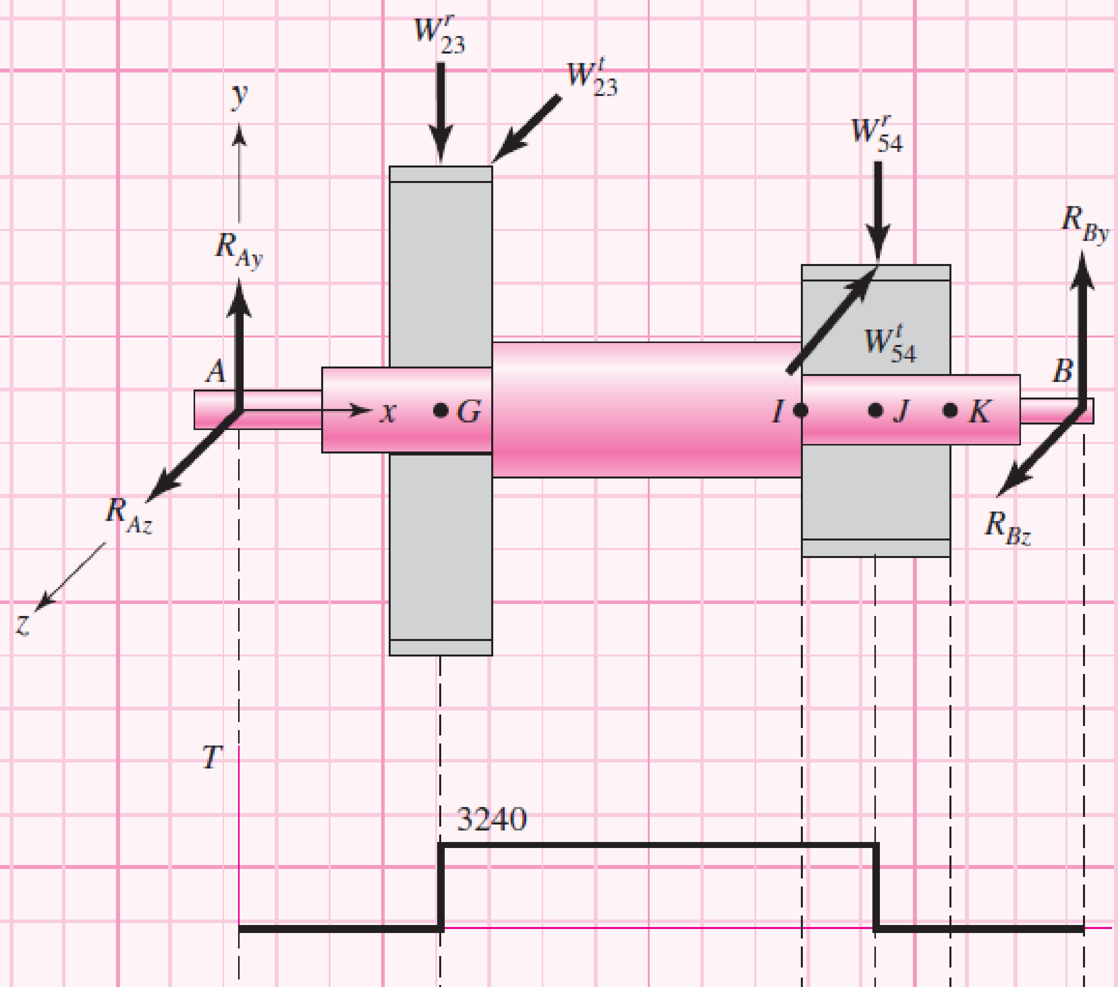
Perform free body diagram analysis to get reaction forces at the bearings.

$R_{Az} = 115.0\,\text{lbf}$

$R_{Ay} = 356.7\,\text{lbf}$

$R_{Bz} = 1776.0\,\text{lbf}$

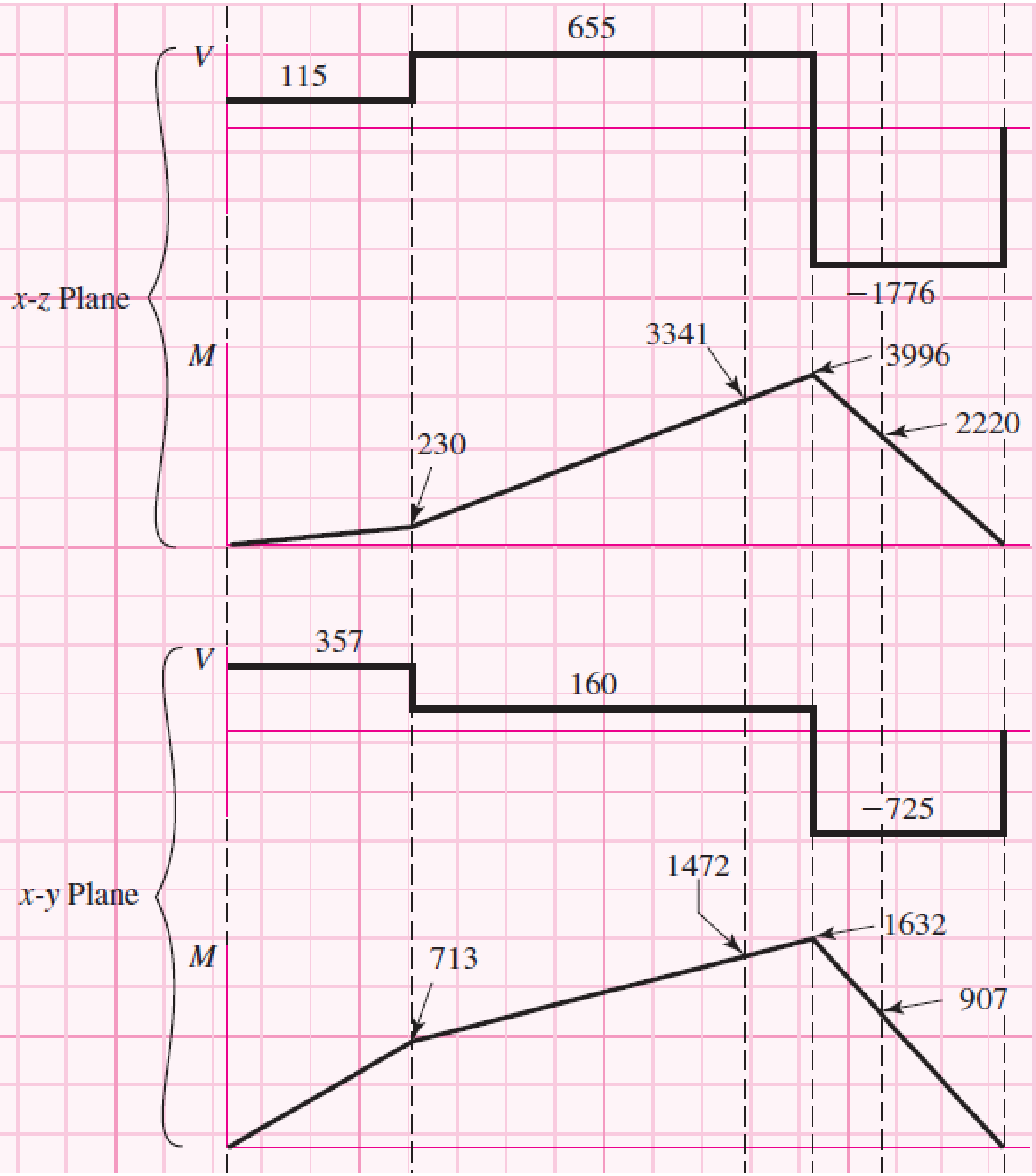
$R_{By} = 725.3\,\text{lbf}$



From ΣM_x, find the torque in the shaft between the gears, $T = W^t_{23}(d_3/2) = 540\,(12/2) = 3240\,\text{lbf} \cdot \text{in.}$

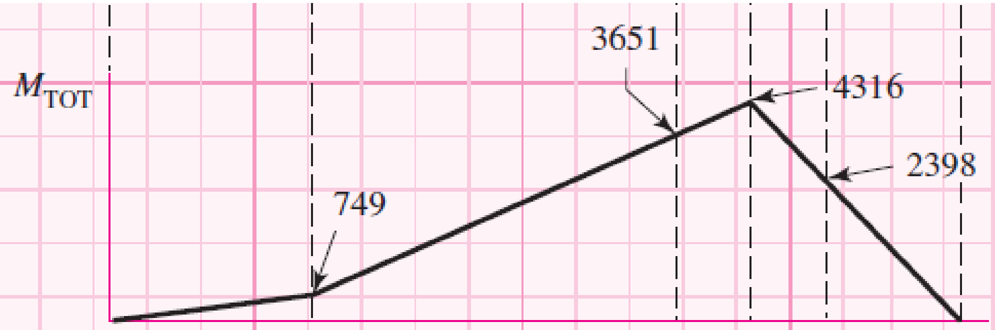
Example 7-2

Generate shear-moment diagrams for two planes.



Example 7-2

Combine orthogonal planes as vectors to get total moments, e.g., at J, $\sqrt{3996^2 + 1632^2} = 4316 \text{ lbf} \cdot \text{in}$.



Start with Point I, where the bending moment is high, there is a stress concentration at the shoulder, and the torque is present.

$$\text{At } I,\ M_a = 3651 \text{ lbf} \cdot \text{in},\ T_m = 3240 \text{ lbf} \cdot \text{in},\ M_m = T_a = 0$$

Assume generous fillet radius for gear at I.

From Table 7-1, estimate $K_t = 1.7$, $K_{ts} = 1.5$. For quick, conservative first pass, assume $K_f = K_t$, $K_{fs} = K_{ts}$.

Example 7-2

Choose inexpensive steel, 1020 CD, with $S_{ut} = 68$ kpsi. For S_e,

Eq. (6–19) $$k_a = aS_{ut}^b = 2.7(68)^{-0.265} = 0.883$$

Guess $k_b = 0.9$. Check later when d is known.

$$k_c = k_d = k_e = 1$$

Eq. (6–18) $$S_e = (0.883)(0.9)(0.5)(68) = 27.0 \text{ kpsi}$$

For first estimate of the small diameter at the shoulder at point I, use the DE-Goodman criterion of Eq. (7-8). This criterion is good for the initial design, since it is simple and conservative. With $M_m = T_a = 0$, Eq. (7-8) reduces to

$$d = \left\{ \frac{16n}{\pi} \left(\frac{2\left(K_f M_a\right)}{S_e} + \frac{\left[3\left(K_{fs} T_m\right)^2\right]^{1/2}}{S_{ut}} \right) \right\}^{1/3}$$

$$d = \left\{ \frac{16(1.5)}{\pi} \left(\frac{2\,(1.7)\,(3651)}{27\,000} + \frac{\{3\,[(1.5)\,(3240)]^2\}^{1/2}}{68\,000} \right) \right\}^{1/3}$$

$$d = 1.65 \text{ in}$$

All estimates have probably been conservative, so select the next standard size below 1.65 in. and check, $d = 1.625$ in.

Example 7-2

A typical D/d ratio for support at a shoulder is $D/d = 1.2$, thus, $D = 1.2(1.625) = 1.95$ in. Increase to $D = 2.0$ in. A nominal 2 in. cold-drawn shaft diameter can be used. Check if estimates were acceptable.

$$D/d = 2/1.625 = 1.23$$

Assume fillet radius $r = d/10 \cong 0.16$ in. $\quad r/d = 0.1$

$$K_t = 1.6 \text{ (Fig. A–15–9)}, q = 0.82 \text{ (Fig. 6–20)}$$

Eq. (6–32) $$K_f = 1 + 0.82(1.6 - 1) = 1.49$$

$$K_{ts} = 1.35 \quad \text{(Fig. A–15–8)}, q_s = 0.85 \text{ (Fig. 6–21)}$$

$$K_{fs} = 1 + 0.85(1.35 - 1) = 1.30$$

$$k_a = 0.883 \text{ (no change)}$$

Eq. (6–20) $$k_b = \left(\frac{1.625}{0.3}\right)^{-0.107} = 0.835$$

$$S_e = (0.883)(0.835)(0.5)(68) = 25.1 \text{ kpsi}$$

Example 7-2

Eq. (7–5) $$\sigma'_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(1.49)(3651)}{\pi(1.625)^3} = 12\,910\text{ psi}$$

Eq. (7–6) $$\sigma'_m = \left[3\left(\frac{16K_{fs}T_m}{\pi d^3}\right)^2\right]^{1/2} = \frac{\sqrt{3}(16)(1.30)(3240)}{\pi(1.625)^3} = 8659\text{ psi}$$

Using Goodman criterion

$$\frac{1}{n_f} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{129\,10}{25\,100} + \frac{8659}{68\,000} = 0.642$$

$$n_f = 1.56$$

Note that we could have used Eq. (7-7) directly.
Check yielding.

$$n_y = \frac{S_y}{\sigma'_{\max}} > \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{57\,000}{12\,910 + 8659} = 2.64$$

Example 7-2

Also check this diameter at the end of the keyway, just to the right of point I, and at the groove at point K. From moment diagram, estimate M at end of keyway to be $M = 3750$ lbf-in.

Assume the radius at the bottom of the keyway will be the standard $r/d = 0.02$, $r = 0.02\ d = 0.02\ (1.625) = 0.0325$ in.

$$K_t = 2.14 \text{ (Table 7–1)},\ q = 0.65 \text{ (Fig. 6–20)}$$

$$K_f = 1 + 0.65(2.14 - 1) = 1.74$$

$$K_{ts} = 3.0 \text{ (Table 7–1)},\ q_s = 0.71 \text{ (Fig. 6–21)}$$

$$K_{fs} = 1 + 0.71(3 - 1) = 2.42$$

$$\sigma'_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(1.74)(3750)}{\pi(1.625)^3} = 15\,490 \text{ psi}$$

$$\sigma'_m = \sqrt{3}(16)\frac{K_{fs}T_m}{\pi d^3} = \frac{\sqrt{3}(16)(2.42)(3240)}{\pi(1.625)^3} = 16\,120 \text{ psi}$$

$$\frac{1}{n_f} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{15\,490}{25\,100} + \frac{16\,120}{68\,000} = 0.854$$

$$n_f = 1.17$$

Example 7-2

The keyway turns out to be more critical than the shoulder. We can either increase the diameter or use a higher strength material. Unless the deflection analysis shows a need for larger diameters, let us choose to increase the strength. We started with a very low strength and can afford to increase it some to avoid larger sizes. Try 1050 CD with $S_{ut} = 100$ kpsi.

Recalculate factors affected by S_{ut}, i.e., $k_a \rightarrow S_e$; $q \rightarrow K_f \rightarrow \sigma'_a$

$$k_a = 2.7(100)^{-0.265} = 0.797, \qquad S_e = 0.797(0.835)(0.5)(100) = 33.3 \text{ kpsi}$$

$$q = 0.72,\ K_f = 1 + 0.72(2.14 - 1) = 1.82$$

$$\sigma'_a = \frac{32(1.82)(3750)}{\pi(1.625)^3} = 16\,200 \text{ psi}$$

$$\frac{1}{n_f} = \frac{16\,200}{33\,300} + \frac{16\,120}{100\,000} = 0.648$$

$$n_f = 1.54$$

Since the Goodman criterion is conservative, we will accept this as close enough to the requested 1.5.

Example 7-2

Check at the groove at K, since K_t for flat-bottomed grooves are often very high. From the torque diagram, note that no torque is present at the groove. From the moment diagram, $M_a = 2398$ lbf · in, $M_m = T_a = T_m = 0$. To quickly check if this location is potentially critical, just use $K_f = K_t = 5.0$ as an estimate, from Table 7-1.

$$\sigma_a = \frac{32 K_f M_a}{\pi d^3} = \frac{32(5)(2398)}{\pi(1.625)^3} = 28\,460 \text{ psi}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{33\,300}{28\,460} = 1.17$$

Example 7-2

This is low. We will look up data for a specific retaining ring to obtain K_f more accurately. With a quick online search of a retaining ring specification using the website www.globalspec.com, appropriate groove specifications for a retaining ring for a shaft diameter of 1.625 in are obtained as follows: width, $a = 0.068$ in; depth, $t = 0.048$ in; and corner radius at bottom of groove, $r = 0.01$ in. From Fig. A–15–16, with $r/t = 0.01/0.048 = 0.208$, and $a/t = 0.068/0.048 = 1.42$

$$K_t = 4.3, q = 0.65 \text{ (Fig. 6–20)}$$

$$K_f = 1 + 0.65(4.3 - 1) = 3.15$$

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(3.15)(2398)}{\pi(1.625)^3} = 17\,930 \text{ psi}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{33\,300}{17\,930} = 1.86$$

Example 7-2

Quickly check if point M might be critical. Only bending is present, and the moment is small, but the diameter is small and the stress concentration is high for a sharp fillet required for a bearing. From the moment diagram,
$M_a = 959$ lbf · in, and $M_m = T_m = T_a = 0$.

Estimate $K_t = 2.7$ from Table 7-1, $d = 1.0$ in, and fillet radius r to fit a typical bearing.

$$r/d = 0.02,\ r = 0.02(1) = 0.02$$

$$q = 0.7 \text{ (Fig. 6–20)}$$

$$K_f = 1 + (0.7)(2.7 - 1) = 2.19$$

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(2.19)(959)}{\pi(1)^3} = 21\,390 \text{ psi}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{33\,300}{21\,390} = 1.56$$

Should be OK. Close enough to recheck after bearing is selected.

Example 7-2

With the diameters specified for the critical locations, fill in trial values for the rest of the diameters, taking into account typical shoulder heights for bearing and gear support.

$$D_1 = D_7 = 1.0 \text{ in}$$

$$D_2 = D_6 = 1.4 \text{ in}$$

$$D_3 = D_5 = 1.625 \text{ in}$$

$$D_4 = 2.0 \text{ in}$$

The bending moments are much less on the left end of shaft, so D_1, D_2, and D_3 could be smaller. However, unless weight is an issue, there is little advantage to requiring more material removal. Also, the extra rigidity may be needed to keep deflections small.

Deflection Considerations

- Deflection analysis at a single point of interest requires complete geometry information for the entire shaft.
- For this reason, a common approach is to size critical locations for stress, then fill in reasonable size estimates for other locations, then perform deflection analysis.
- Deflection of the shaft, both linear and angular, should be checked at gears and bearings.

Deflection Considerations

- Allowable deflections at components will depend on the component manufacturer's specifications.
- Typical ranges are given in Table 7–2

Table 7–2

Typical Maximum Ranges for Slopes and Transverse Deflections

Slopes	
Tapered roller	0.0005–0.0012 rad
Cylindrical roller	0.0008–0.0012 rad
Deep-groove ball	0.001–0.003 rad
Spherical ball	0.026–0.052 rad
Self-align ball	0.026–0.052 rad
Uncrowned spur gear	< 0.0005 rad
Transverse Deflections	
Spur gears with $P < 10$ teeth/in	0.010 in
Spur gears with $11 < P < 19$	0.005 in
Spur gears with $20 < P < 50$	0.003 in

Deflection Considerations

- Deflection analysis is straightforward, but lengthy and tedious to carry out manually.
- Each point of interest requires entirely new deflection analysis.
- Consequently, shaft deflection analysis is almost always done with the assistance of software.
- Options include specialized shaft software, general beam deflection software, and finite element analysis software.

Example 7-3

This example problem is part of a larger case study. See Chap. 18 for the full context.

In Ex. 7-2, a preliminary shaft geometry was obtained on the basis of design for stress. The resulting shaft is shown in Fig. 7-10, with proposed diameters of

$$D_1 = D_7 = 1 \text{ in}$$

$$D_2 = D_6 = 1.4 \text{ in}$$

$$D_3 = D_5 = 1.625 \text{ in}$$

$$D_4 = 2.0 \text{ in}$$

Check that the deflections and slopes at the gears and bearings are acceptable. If necessary, propose changes in the geometry to resolve any problems.

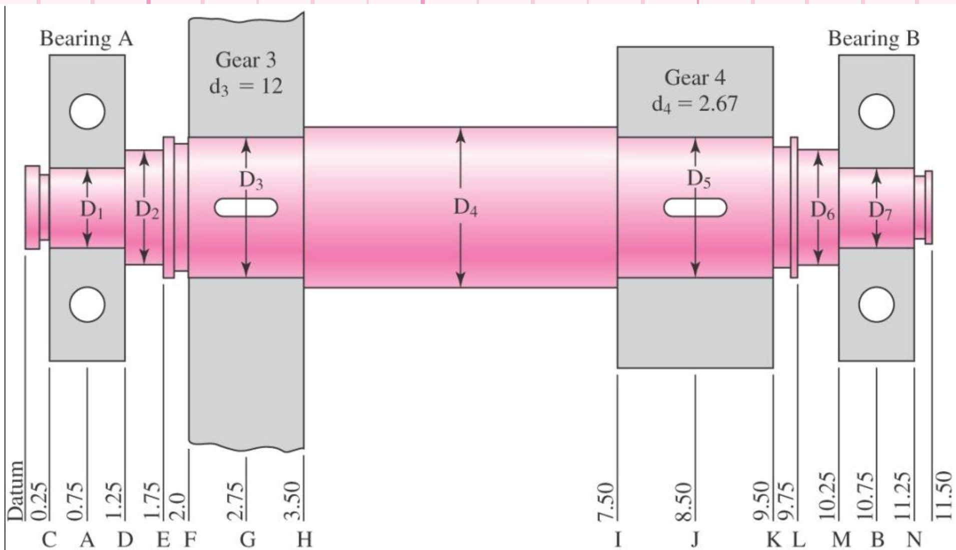


Fig. 7-10

Example 7-3

Solution

A simple planar beam analysis program will be used. By modeling the shaft twice, with loads in two orthogonal planes, and combining the results, the shaft deflections can readily be obtained. For both planes, the material is selected (steel with $E = 30$ Mpsi), the shaft lengths and diameters are entered, and the bearing locations are specified. Local details like grooves and keyways are ignored, as they will have insignificant effect on the deflections. Then the tangential gear forces are entered in the horizontal *xz* plane model, and the radial gear forces are entered in the vertical *xy* plane model. The software can calculate the bearing reaction forces, and numerically integrate to generate plots for shear, moment, slope, and deflection, as shown in Fig. 7–11.

Example 7-3

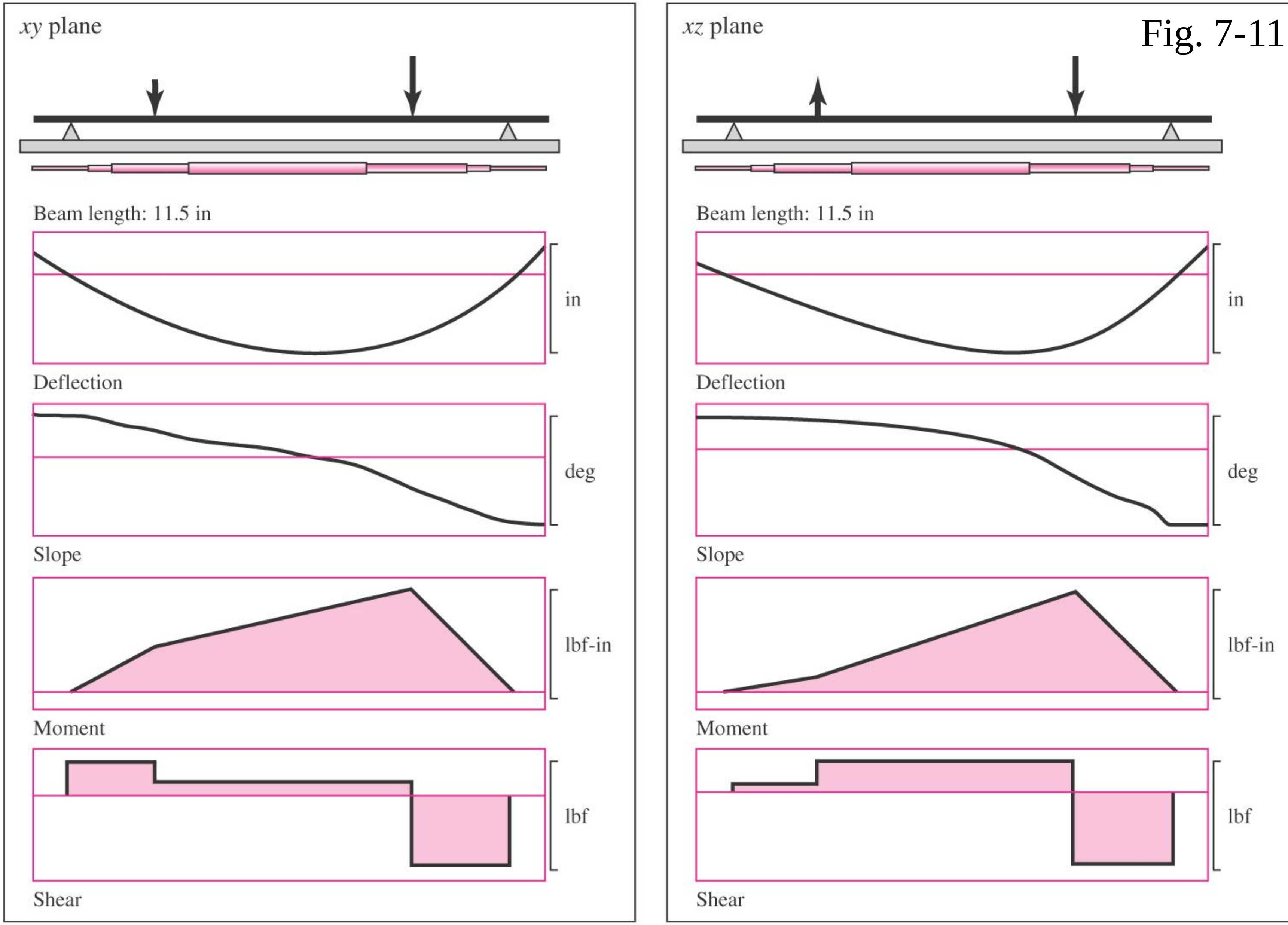


Fig. 7-11

Example 7-3

The deflections and slopes at points of interest are obtained from the plots, and combined with orthogonal vector addition, that is, $\delta = \sqrt{\delta_{xz}^2 + \delta_{xy}^2}$. Results are shown in Table 7-3.

Point of Interest	xz Plane	xy Plane	Total
Left bearing slope	0.02263 deg	0.01770 deg	0.02872 deg
			0.000501 rad
Right bearing slope	0.05711 deg	0.02599 deg	0.06274 deg
			0.001095 rad
Left gear slope	0.02067 deg	0.01162 deg	0.02371 deg
			0.000414 rad
Right gear slope	0.02155 deg	0.01149 deg	0.02442 deg
			0.000426 rad
Left gear deflection	0.0007568 in	0.0005153 in	0.0009155 in
Right gear deflection	0.0015870 in	0.0007535 in	0.0017567 in

Table 7–3

Example 7-3

Whether these values are acceptable will depend on the specific bearings and gears selected, as well as the level of performance expected. According to the guidelines in Table 7–2, all of the bearing slopes are well below typical limits for ball bearings. The right bearing slope is within the typical range for cylindrical bearings. Since the load on the right bearing is relatively high, a cylindrical bearing might be used. This constraint should be checked against the specific bearing specifications once the bearing is selected.

The gear slopes and deflections more than satisfy the limits recommended in Table 7–2. It is recommended to proceed with the design, with an awareness that changes that reduce rigidity should warrant another deflection check.

Adjusting Diameters for Allowable Deflections

- If any deflection is larger than allowed, since I is proportional to d^4, a new diameter can be found from

$$d_{\text{new}} = d_{\text{old}} \left| \frac{n_d\, y_{\text{old}}}{y_{\text{all}}} \right|^{1/4} \tag{7–17}$$

- Similarly, for slopes,

$$d_{\text{new}} = d_{\text{old}} \left| \frac{n_d (dy/dx)_{\text{old}}}{(\text{slope})_{\text{all}}} \right|^{1/4} \tag{7–18}$$

- Determine the largest d_{new}/d_{old} ratio, then multiply all diameters by this ratio.
- The tight constraint will be just right, and the others will be loose.

Example 7-4

For the shaft in Ex. 7–3, it was noted that the slope at the right bearing is near the limit for a cylindrical roller bearing. Determine an appropriate increase in diameters to bring this slope down to 0.0005 rad.

Solution

Applying Eq. (7–17) to the deflection at the right bearing gives

$$d_{\text{new}} = d_{\text{old}} \left| \frac{n_d \text{slope}_{\text{old}}}{\text{slope}_{\text{all}}} \right|^{1/4} = 1.0 \left| \frac{(1)(0.001095)}{(0.0005)} \right|^{1/4} = 1.216 \text{ in}$$

Multiplying all diameters by the ratio

$$\frac{d_{\text{new}}}{d_{\text{old}}} = \frac{1.216}{1.0} = 1.216$$

gives a new set of diameters,

$$D_1 = D_7 = 1.216 \text{ in}$$

$$D_2 = D_6 = 1.702 \text{ in}$$

$$D_3 = D_5 = 1.976 \text{ in}$$

$$D_4 = 2.432 \text{ in}$$

Repeating the beam deflection analysis of Ex. 7–3 with these new diameters produces a slope at the right bearing of 0.0005 in, with all other deflections less than their previous values.

Angular Deflection of Shafts

- For stepped shaft with individual cylinder length l_i and torque T_i, the angular deflection can be estimated from

$$\theta = \sum \theta_i = \sum \frac{T_i l_i}{G_i J_i} \qquad (7\text{–}19)$$

- For constant torque throughout homogeneous material

$$\theta = \frac{T}{G} \sum \frac{l_i}{J_i} \qquad (7\text{–}20)$$

- Experimental evidence shows that these equations slightly underestimate the angular deflection.
- Torsional stiffness of a stepped shaft is

$$\frac{1}{k} = \sum \frac{1}{k_i} \qquad (7\text{–}21)$$

Critical Speeds for Shafts

- A shaft with mass has a critical speed at which its deflections become unstable.
- Components attached to the shaft have an even lower critical speed than the shaft.
- Designers should ensure that the lowest critical speed is at least twice the operating speed.

Critical Speeds for Shafts

- For a simply supported shaft of uniform diameter, the first critical speed is

$$\omega_1 = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{EI}{m}} = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{gEI}{A\gamma}} \qquad (7\text{–}22)$$

- For an ensemble of attachments, Rayleigh's method for lumped masses gives

$$\omega_1 = \sqrt{\frac{g \sum w_i y_i}{\sum w_i y_i^2}} \qquad (7\text{–}23)$$

Critical Speeds for Shafts

- Eq. (7–23) can be applied to the shaft itself by partitioning the shaft into segments.

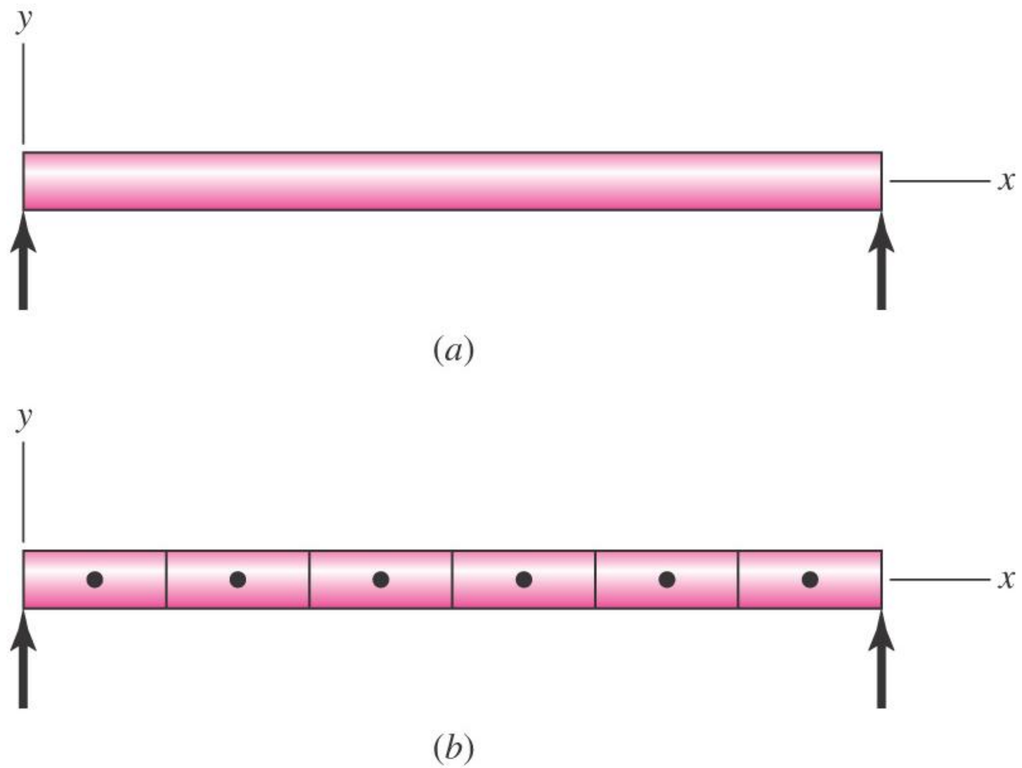


Fig. 7–12

Critical Speeds for Shafts

- An *influence coefficient* is the transverse deflection at location *I* due to a unit load at location *j*.
- From Table A–9–6 for a simply supported beam with a single unit load

$$\delta_{ij} = \begin{cases} \dfrac{b_j x_i}{6EIl}\left(l^2 - b_j^2 - x_i^2\right) & x_i \le a_i \\ \dfrac{a_j(l - x_i)}{6EIl}\left(2l x_i - a_j^2 - x_i^2\right) & x_i > a_i \end{cases} \qquad (7\text{–}24)$$

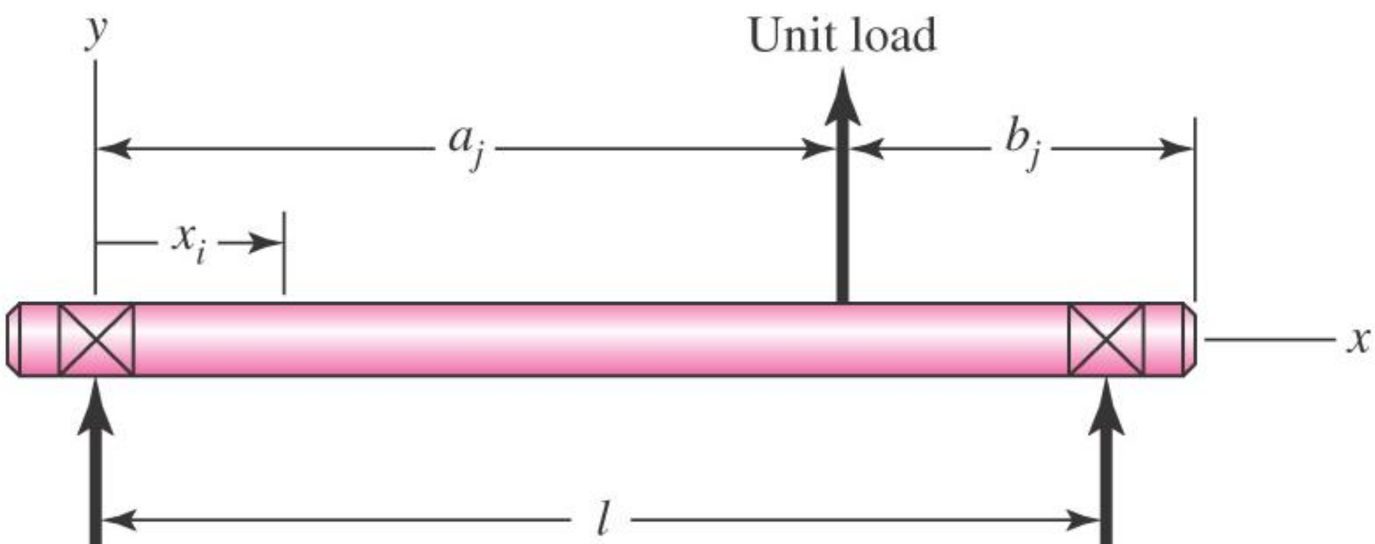


Fig. 7–13

Critical Speeds for Shafts

- Taking for example a simply supported shaft with three loads, the deflections corresponding to the location of each load is

$$
\begin{aligned}
y_1 &= F_1\delta_{11} + F_2\delta_{12} + F_3\delta_{13} \\
y_2 &= F_1\delta_{21} + F_2\delta_{22} + F_3\delta_{23} \\
y_3 &= F_1\delta_{31} + F_2\delta_{32} + F_3\delta_{33}
\end{aligned}
\qquad (7\text{–}25)
$$

- If the forces are due only to centrifugal force due to the shaft mass,

$$
\begin{aligned}
y_1 &= m_1\omega^2 y_1\delta_{11} + m_2\omega^2 y_2\delta_{12} + m_3\omega^2 y_3\delta_{13} \\
y_2 &= m_1\omega^2 y_1\delta_{21} + m_2\omega^2 y_2\delta_{22} + m_3\omega^2 y_3\delta_{23} \\
y_3 &= m_1\omega^2 y_1\delta_{31} + m_2\omega^2 y_2\delta_{32} + m_3\omega^2 y_3\delta_{33}
\end{aligned}
$$

- Rearranging,

$$
\begin{aligned}
(m_1\delta_{11} - 1/\omega^2)y_1 + (m_2\delta_{12})y_2 + (m_3\delta_{13})y_3 &= 0 \\
(m_1\delta_{21})y_1 + (m_2\delta_{22} - 1/\omega^2)y_2 + (m_3\delta_{23})y_3 &= 0 \\
(m_1\delta_{31})y_1 + (m_2\delta_{32})y_2 + (m_3\delta_{33} - 1/\omega^2)y_3 &= 0
\end{aligned}
$$

Critical Speeds for Shafts

- Non-trivial solutions to this set of simultaneous equations will exist when its determinant equals zero.

$$\begin{vmatrix} (m_1\delta_{11} - 1/\omega^2) & m_2\delta_{12} & m_3\delta_{13} \\ m_1\delta_{21} & (m_2\delta_{22} - 1/\omega^2) & m_3\delta_{23} \\ m_1\delta_{31} & m_2\delta_{32} & (m_3\delta_{33} - 1/\omega^2) \end{vmatrix} = 0 \qquad (7\text{–}26)$$

- Expanding the determinant,

$$\left(\frac{1}{\omega^2}\right)^3 - (m_1\delta_{11} + m_2\delta_{22} + m_3\delta_{33})\left(\frac{1}{\omega^2}\right)^2 + \cdots = 0 \qquad (7\text{–}27)$$

- Eq. (7–27) can be written in terms of its three roots as

$$\left(\frac{1}{\omega^2} - \frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega^2} - \frac{1}{\omega_2^2}\right)\left(\frac{1}{\omega^2} - \frac{1}{\omega_3^2}\right) = 0$$

or

$$\left(\frac{1}{\omega^2}\right)^3 - \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2}\right)\left(\frac{1}{\omega^2}\right)^2 + \cdots = 0 \qquad (7\text{–}28)$$

Critical Speeds for Shafts

- Comparing Eqs. (7–27) and (7–28),

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} = m_1\delta_{11} + m_2\delta_{22} + m_3\delta_{33} \qquad (7\text{–}29)$$

- Define ω_{ii} as the critical speed if m_i is acting alone. From Eq. (7–29),

$$\frac{1}{\omega_{ii}^2} = m_i\delta_{ii}$$

- Thus, Eq. (7–29) can be rewritten as

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2} \qquad (7\text{–}30)$$

Critical Speeds for Shafts

- Note that $1/\omega_1^2 \gg 1/\omega_2^2$, and $1/\omega_3^2$
- The first critical speed can be approximated from Eq. (7–30) as

$$\frac{1}{\omega_1^2} \doteq \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2} \qquad (7\text{–}31)$$

- Extending this idea to an *n*-body shaft, we obtain *Dunkerley's equation,*

$$\frac{1}{\omega_1^2} \doteq \sum_{1=1}^{n} \frac{1}{\omega_{ii}^2} \qquad (7\text{–}32)$$

Critical Speeds for Shafts

- Since Dunkerley's equation has loads appearing in the equation, it follows that if each load could be placed at some convenient location transformed into an equivalent load, then the critical speed of an array of loads could be found by summing the equivalent loads, all placed at a single convenient location.
- For the load at station 1, placed at the center of the span, the equivalent load is found from

$$\omega_{11}^2 = \frac{1}{m_1 \delta_{11}} = \frac{g}{w_1 \delta_{11}} = \frac{g}{w_{1c} \delta_{cc}}$$

or

$$w_{1c} = w_1 \frac{\delta_{11}}{\delta_{cc}} \tag{7–33}$$

Example 7-5

Consider a simply supported steel shaft as depicted in Fig. 7–14, with 1 in diameter and a 31-in span between bearings, carrying two gears weighing 35 and 55 lbf.

(*a*) Find the influence coefficients.

(*b*) Find $\sum wy$ and $\sum wy^2$ and the first critical speed using Rayleigh's equation, Eq. (7–23).

(*c*) From the influence coefficients, find ω_{11} and ω_{22}.

(*d*) Using Dunkerley's equation, Eq. (7–32), estimate the first critical speed.

(*e*) Use superposition to estimate the first critical speed.

(*f*) Estimate the shaft's intrinsic critical speed. Suggest a modification to Dunkerley's equation to include the effect of the shaft's mass on the first critical speed of the attachments.

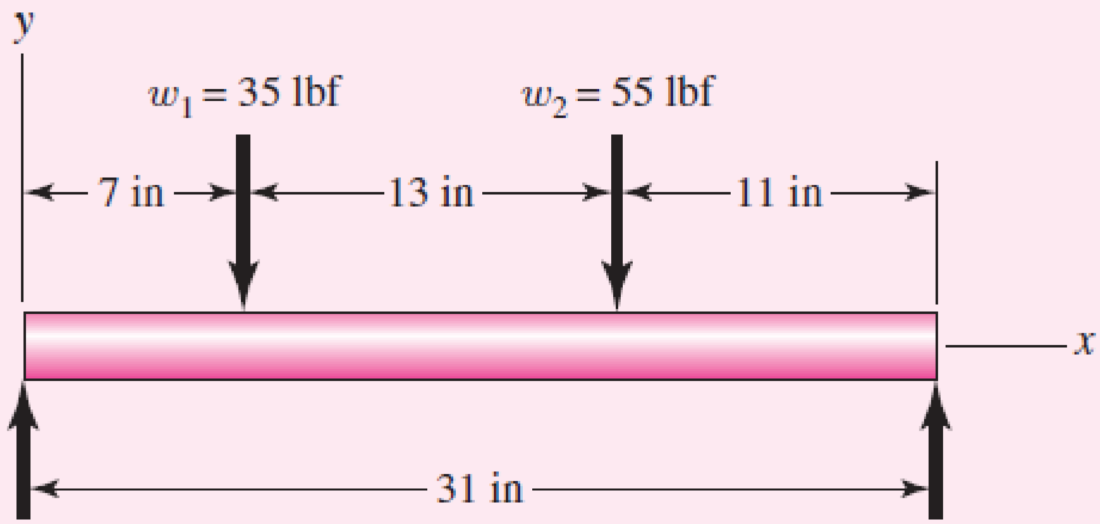


Fig. 7–14 (*a*)

Example 7-5

(*a*)
$$I = \frac{\pi d^4}{64} = \frac{\pi (1)^4}{64} = 0.049\,09 \text{ in}^4$$

$$6EIl = 6(30)10^6(0.049\,09)31 = 0.2739(10^9) \text{ lbf}\cdot\text{in}^3$$

From Eq. set (7–24),

$$\delta_{11} = \frac{24(7)(31^2 - 24^2 - 7^2)}{0.2739(10^9)} = 2.061(10^{-4}) \text{ in/lbf}$$

$$\delta_{22} = \frac{11(20)(31^2 - 11^2 - 20^2)}{0.2739(10^9)} = 3.534(10^{-4}) \text{ in/lbf}$$

$$\delta_{12} = \delta_{21} = \frac{11(7)(31^2 - 11^2 - 7^2)}{0.2739(10^9)} = 2.224(10^{-4}) \text{ in/lbf}$$

$$y_1 = w_1\delta_{11} + w_2\delta_{12} = 35(2.061)10^{-4} + 55(2.224)10^{-4} = 0.019\,45 \text{ in}$$

$$y_2 = w_1\delta_{21} + w_2\delta_{22} = 35(2.224)10^{-4} + 55(3.534)10^{-4} = 0.027\,22 \text{ in}$$

Example 7-5

(*b*)

$$\sum w_i y_i = 35(0.019\,45) + 55(0.027\,22) = 2.178 \text{ lbf} \cdot \text{in}$$

$$\sum w_i y_i^2 = 35(0.019\,45)^2 + 55(0.027\,22)^2 = 0.053\,99 \text{ lbf} \cdot \text{in}^2$$

$$\omega = \sqrt{\frac{386.1(2.178)}{0.053\,99}} = 124.8 \text{ rad/s, or } 1192 \text{ rev/min}$$

(*c*)

$$\frac{1}{\omega_{11}^2} = \frac{w_1}{g}\delta_{11}$$

$$\omega_{11} = \sqrt{\frac{g}{w_1\delta_{11}}} = \sqrt{\frac{386.1}{35(2.061)10^{-4}}} = 231.4 \text{ rad/s, or } 2210 \text{ rev/min}$$

$$\omega_{22} = \sqrt{\frac{g}{w_2\delta_{22}}} = \sqrt{\frac{386.1}{55(3.534)10^{-4}}} = 140.9 \text{ rad/s, or } 1346 \text{ rev/min}$$

Example 7-5

(*d*)
$$\frac{1}{\omega_1^2} \doteq \sum \frac{1}{\omega_{ii}^2} = \frac{1}{231.4^2} + \frac{1}{140.9^2} = 6.905(10^{-5}) \qquad (1)$$

$$\omega_1 \doteq \sqrt{\frac{1}{6.905(10^{-5})}} = 120.3 \text{ rad/s, or } 1149 \text{ rev/min}$$

which is less than part *b*, as expected.
(*e*) From Eq. (7–24),

$$\delta_{cc} = \frac{b_{cc} x_{cc}\left(l^2 - b_{cc}^2 - x_{cc}^2\right)}{6EIl} = \frac{15.5(15.5)(31^2 - 15.5^2 - 15.5^2)}{0.2739(10^9)}$$
$$= 4.215(10^{-4}) \text{ in/lbf}$$

Example 7-5

From Eq. (7–33),

$$w_{1c} = w_1 \frac{\delta_{11}}{\delta_{cc}} = 35 \frac{2.061(10^{-4})}{4.215(10^{-4})} = 17.11 \text{ lbf}$$

$$w_{2c} = w_2 \frac{\delta_{22}}{\delta_{cc}} = 55 \frac{3.534(10^{-4})}{4.215(10^{-4})} = 46.11 \text{ lbf}$$

$$\omega = \sqrt{\frac{g}{\delta_{cc} \sum w_{ic}}} = \sqrt{\frac{386.1}{4.215(10^{-4})(17.11 + 46.11)}} = 120.4 \text{ rad/s, or } 1150 \text{ rev/min}$$

which, except for rounding, agrees with part *d*, as expected.

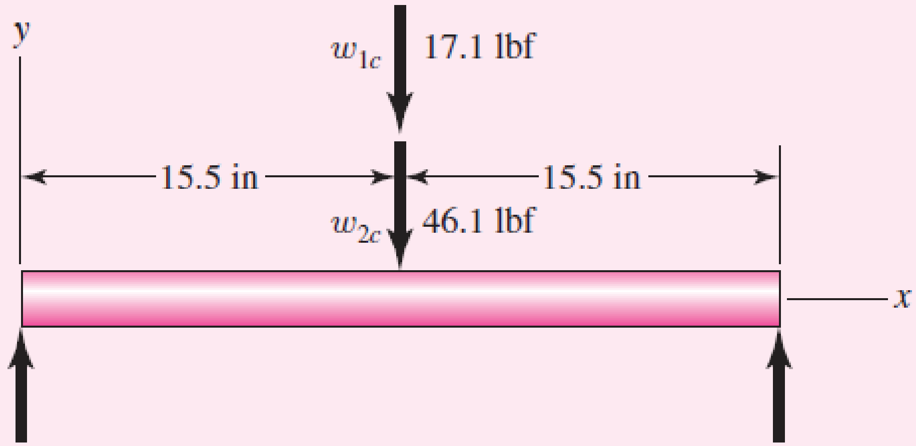


Fig. 7–14 (*b*)

Example 7-5

(f) For the shaft, $E = 30(10^6)$ psi, $\gamma = 0.282$ lbf/in^3, and $A = \pi(1^2)/4 = 0.7854$ in^2. Considering the shaft alone, the critical speed, from Eq. (7–22), is

$$\omega_s = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{gEI}{A\gamma}} = \left(\frac{\pi}{31}\right)^2 \sqrt{\frac{386.1(30)10^6(0.049\,09)}{0.7854(0.282)}}$$

$$= 520.4 \text{ rad/s, or } 4970 \text{ rev/min}$$

We can simply add $1/\omega_s^2$ to the right side of Dunkerley's equation, Eq. (1), to include the shaft's contribution,

$$\frac{1}{\omega_1^2} \doteq \frac{1}{520.4^2} + 6.905(10^{-5}) = 7.274(10^{-5})$$

$$\omega_1 \doteq 117.3 \text{ rad/s, or } 1120 \text{ rev/min}$$

which is slightly less than part d, as expected.

Example 7-5

The shaft's first critical speed ω_s is just one more single effect to add to Dunkerley's equation. Since it does not fit into the summation, it is usually written up front.

$$\frac{1}{\omega_1^2} \doteq \frac{1}{\omega_s^2} + \sum_{i=1}^{n} \frac{1}{\omega_{ii}^2} \tag{7–34}$$

Common shafts are complicated by the stepped-cylinder geometry, which makes the influence-coefficient determination part of a numerical solution.

Setscrews

- Setscrews resist axial and rotational motion
- They apply a compressive force to create friction
- The tip of the set screw may also provide a slight penetration
- Various tips are available

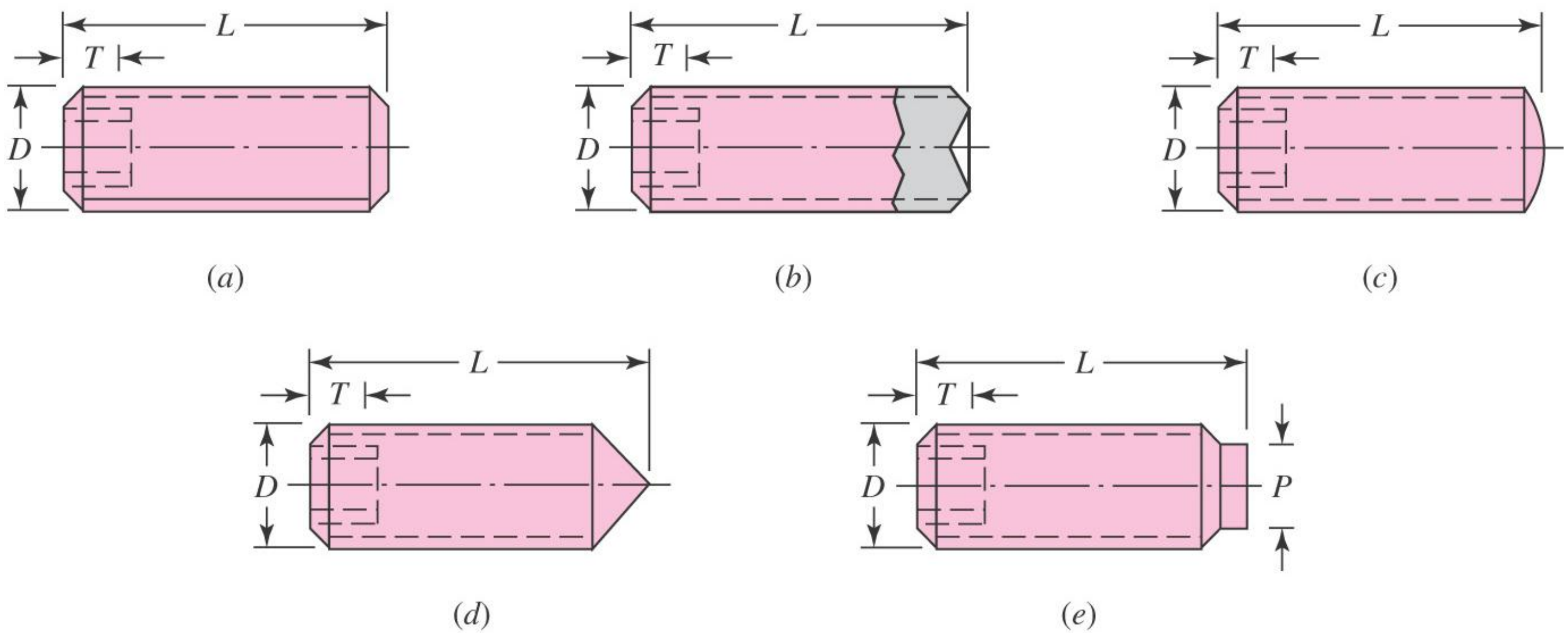


Fig. 7–15

Setscrews

- Resistance to axial motion of collar or hub relative to shaft is called *holding power*
- Typical values listed in Table 7–4 apply to axial and torsional resistance
- Typical factors of safety are 1.5 to 2.0 for static, and 4 to 8 for dynamic loads
- Length should be about half the shaft diameter

Table 7–4

Typical Holding Power (Force) for Socket Setscrews*

Source: Unbrako Division, SPS Technologies, Jenkintown, Pa.

Size, in	Seating Torque, lbf · in	Holding Power, lbf
#0	1.0	50
#1	1.8	65
#2	1.8	85
#3	5	120
#4	5	160
#5	10	200
#6	10	250
#8	20	385
#10	36	540
$\frac{1}{4}$	87	1000
$\frac{5}{16}$	165	1500
$\frac{3}{8}$	290	2000
$\frac{7}{16}$	430	2500
$\frac{1}{2}$	620	3000
$\frac{9}{16}$	620	3500
$\frac{5}{8}$	1325	4000
$\frac{3}{4}$	2400	5000
$\frac{7}{8}$	5200	6000
1	7200	7000

Keys and Pins

- Used to secure rotating elements and to transmit torque

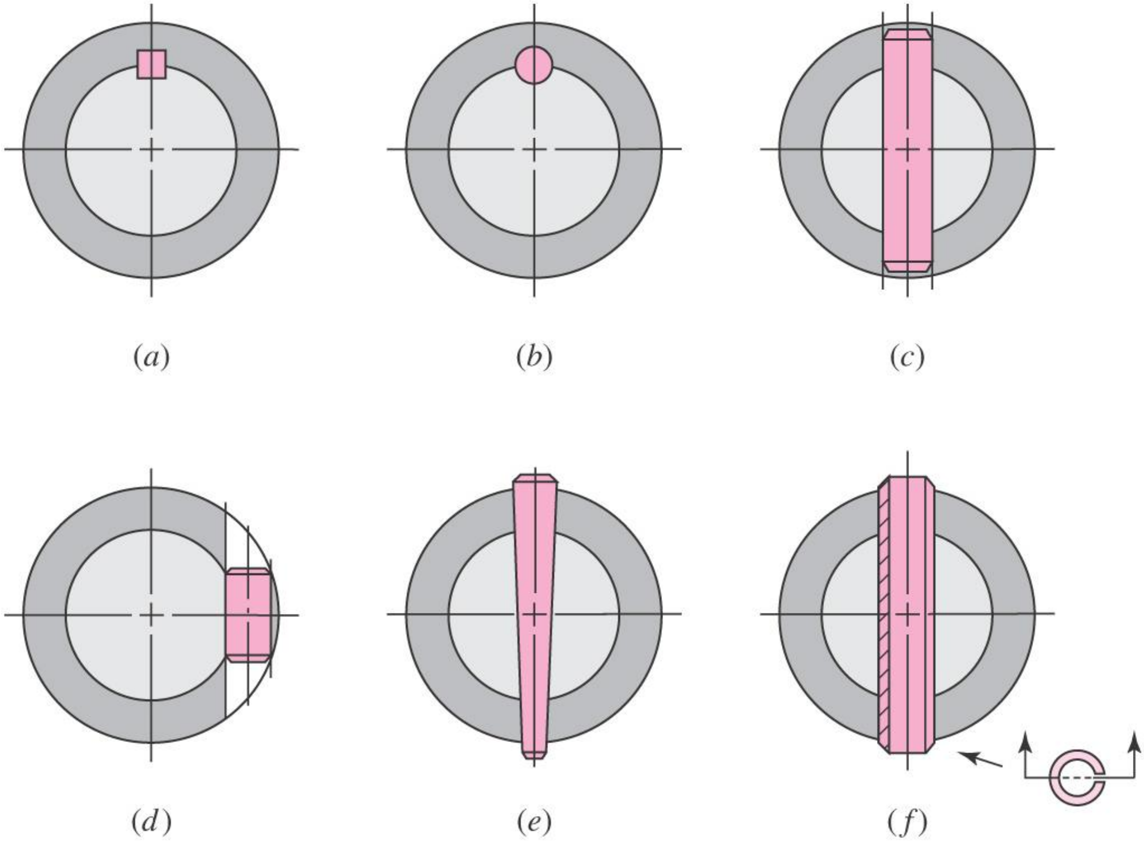


Fig. 7–16

Tapered Pins

- Taper pins are sized by diameter at large end
- Small end diameter is

$$d = D - 0.0208L \tag{7–35}$$

- Table 7–5 shows some standard sizes in inches

Size	Commercial Maximum	Commercial Minimum	Precision Maximum	Precision Minimum
4/0	0.1103	0.1083	0.1100	0.1090
2/0	0.1423	0.1403	0.1420	0.1410
0	0.1573	0.1553	0.1570	0.1560
2	0.1943	0.1923	0.1940	0.1930
4	0.2513	0.2493	0.2510	0.2500
6	0.3423	0.3403	0.3420	0.3410
8	0.4933	0.4913	0.4930	0.4920

Table 7–5

Keys

- Keys come in standard square and rectangular sizes
- Shaft diameter determines key size

Shaft Diameter Over	Shaft Diameter To (Incl.)	Key Size *w*	Key Size *h*	Keyway Depth
5/16	7/16	3/32	3/32	3/64
7/16	9/16	1/8	3/32	3/64
		1/8	1/8	1/16
9/16	7/8	3/16	1/8	1/16
		3/16	3/16	3/32
7/8	1 1/4	1/4	3/16	3/32
		1/4	1/4	1/8
1 1/4	1 3/8	5/16	1/4	1/8
		5/16	5/16	5/32
1 3/8	1 3/4	3/8	1/4	1/8
		3/8	3/8	3/16
1 3/4	2 1/4	1/2	3/8	3/16
		1/2	1/2	1/4
2 1/4	2 3/4	5/8	7/16	7/32
		5/8	5/8	5/16
2 3/4	3 1/4	3/4	1/2	1/4
		3/4	3/4	3/8

Table 7–6

Keys

- Failure of keys is by either direct shear or bearing stress
- Key length is designed to provide desired factor of safety
- Factor of safety should not be excessive, so the inexpensive key is the weak link
- Key length is limited to hub length
- Key length should not exceed 1.5 times shaft diameter to avoid problems from twisting
- Multiple keys may be used to carry greater torque, typically oriented 90º from one another
- Stock key material is typically low carbon cold-rolled steel, with dimensions slightly under the nominal dimensions to easily fit end-milled keyway
- A setscrew is sometimes used with a key for axial positioning, and to minimize rotational backlash

Gib-head Key

- Gib-head key is tapered so that when firmly driven it prevents axial motion
- Head makes removal easy
- Projection of head may be hazardous

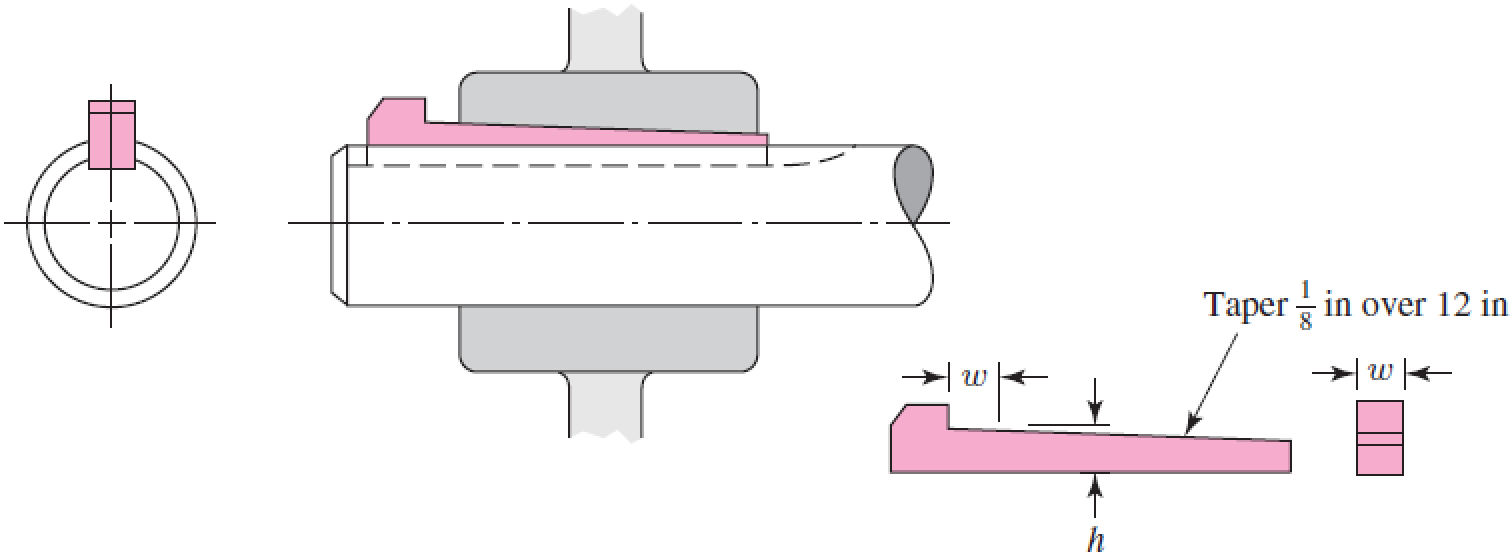


Fig. 7–17 (*a*)

Woodruff Key

- Woodruff keys have deeper penetration
- Useful for smaller shafts to prevent key from rolling
- When used near a shoulder, the keyway stress concentration interferes less with shoulder than square keyway

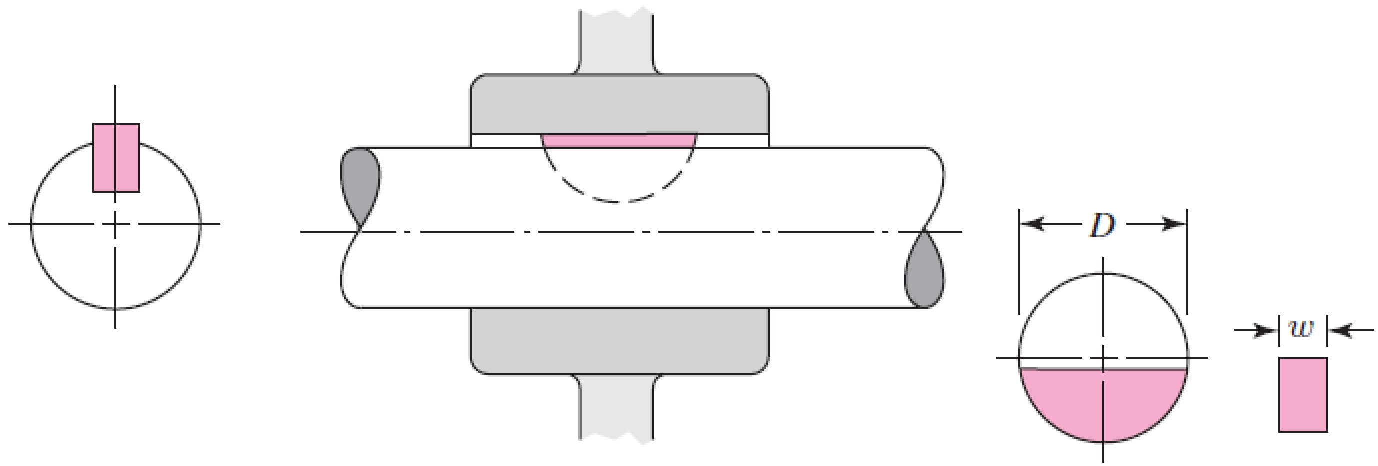


Fig. 7–17 (b)

Woodruff Key

Table 7–7

Dimensions of Woodruff Keys—Inch Series

Key Size w	Key Size D	Height b	Offset e	Keyseat Depth Shaft	Keyseat Depth Hub
$\frac{1}{16}$	$\frac{1}{4}$	0.109	$\frac{1}{64}$	0.0728	0.0372
$\frac{1}{16}$	$\frac{3}{8}$	0.172	$\frac{1}{64}$	0.1358	0.0372
$\frac{3}{32}$	$\frac{3}{8}$	0.172	$\frac{1}{64}$	0.1202	0.0529
$\frac{3}{32}$	$\frac{1}{2}$	0.203	$\frac{3}{64}$	0.1511	0.0529
$\frac{3}{32}$	$\frac{5}{8}$	0.250	$\frac{1}{16}$	0.1981	0.0529
$\frac{1}{8}$	$\frac{1}{2}$	0.203	$\frac{3}{64}$	0.1355	0.0685
$\frac{1}{8}$	$\frac{5}{8}$	0.250	$\frac{1}{16}$	0.1825	0.0685
$\frac{1}{8}$	$\frac{3}{4}$	0.313	$\frac{1}{16}$	0.2455	0.0685
$\frac{5}{32}$	$\frac{5}{8}$	0.250	$\frac{1}{16}$	0.1669	0.0841
$\frac{5}{32}$	$\frac{3}{4}$	0.313	$\frac{1}{16}$	0.2299	0.0841
$\frac{5}{32}$	$\frac{7}{8}$	0.375	$\frac{1}{16}$	0.2919	0.0841
$\frac{3}{16}$	$\frac{3}{4}$	0.313	$\frac{1}{16}$	0.2143	0.0997
$\frac{3}{16}$	$\frac{7}{8}$	0.375	$\frac{1}{16}$	0.2763	0.0997
$\frac{3}{16}$	1	0.438	$\frac{1}{16}$	0.3393	0.0997
$\frac{1}{4}$	$\frac{7}{8}$	0.375	$\frac{1}{16}$	0.2450	0.1310
$\frac{1}{4}$	1	0.438	$\frac{1}{16}$	0.3080	0.1310
$\frac{1}{4}$	$1\frac{1}{4}$	0.547	$\frac{5}{64}$	0.4170	0.1310
$\frac{5}{16}$	1	0.438	$\frac{1}{16}$	0.2768	0.1622
$\frac{5}{16}$	$1\frac{1}{4}$	0.547	$\frac{5}{64}$	0.3858	0.1622
$\frac{5}{16}$	$1\frac{1}{2}$	0.641	$\frac{7}{64}$	0.4798	0.1622
$\frac{3}{8}$	$1\frac{1}{4}$	0.547	$\frac{5}{64}$	0.3545	0.1935
$\frac{3}{8}$	$1\frac{1}{2}$	0.641	$\frac{7}{64}$	0.4485	0.1935

Woodruff Key

Table 7–8

Sizes of Woodruff Keys Suitable for Various Shaft Diameters

Keyseat Width, in	Shaft Diameter, in From	Shaft Diameter, in To (inclusive)
$\frac{1}{16}$	$\frac{5}{16}$	$\frac{1}{2}$
$\frac{3}{32}$	$\frac{3}{8}$	$\frac{7}{8}$
$\frac{1}{8}$	$\frac{3}{8}$	$1\frac{1}{2}$
$\frac{5}{32}$	$\frac{1}{2}$	$1\frac{5}{8}$
$\frac{3}{16}$	$\frac{9}{16}$	2
$\frac{1}{4}$	$\frac{11}{16}$	$2\frac{1}{4}$
$\frac{5}{16}$	$\frac{3}{4}$	$2\frac{3}{8}$
$\frac{3}{8}$	1	$2\frac{5}{8}$

Stress Concentration Factors for Keys

- For keyseats cut by standard end-mill cutters, with a ratio of $r/d = 0.02$, Peterson's charts give
  - $K_t = 2.14$ for bending
  - $K_t = 2.62$ for torsion without the key in place
  - $K_t = 3.0$ for torsion with the key in place
- Keeping the end of the keyseat at least a distance of $d/10$ from the shoulder fillet will prevent the two stress concentrations from combining.

Example 7-6

A UNS G10350 steel shaft, heat-treated to a minimum yield strength of 75 kpsi, has a diameter of $1\frac{7}{16}$ in. The shaft rotates at 600 rev/min and transmits 40 hp through a gear. Select an appropriate key for the gear.

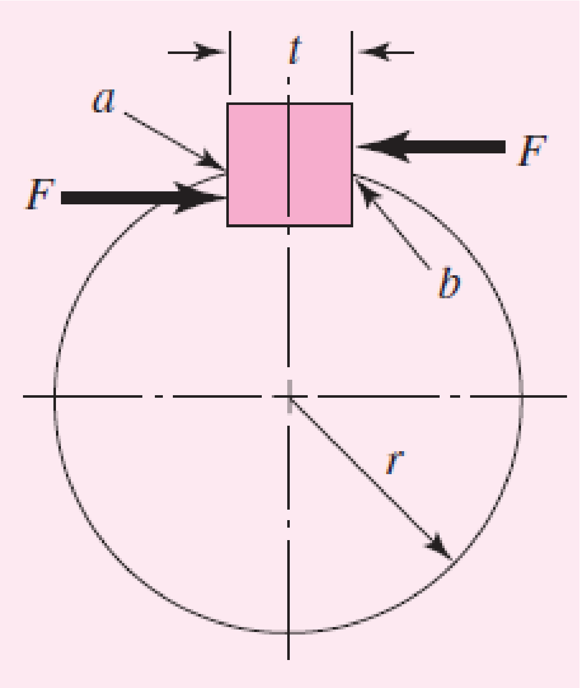


Fig. 7–19

Example 7-6

A $\frac{3}{8}$-in square key is selected, UNS G10200 cold-drawn steel being used. The design will be based on a yield strength of 65 kpsi. A factor of safety of 2.80 will be employed in the absence of exact information about the nature of the load.

The torque is obtained from the horsepower equation

$$T = \frac{63\,025H}{n} = \frac{(63\,025)(40)}{600} = 4200\,\text{lbf} \cdot \text{in}$$

From Fig. 7–19, the force F at the surface of the shaft is

$$F = \frac{T}{r} = \frac{4200}{1.4375/2} = 5850\,\text{lbf}$$

By the distortion-energy theory, the shear strength is

$$S_{sy} = 0.577S_y = (0.577)(65) = 37.5\,\text{kpsi}$$

Example 7-6

Failure by shear across the area *ab* will create a stress of $\tau = F/tl$. Substituting the strength divided by the factor of safety for τ gives

$$\frac{S_{sy}}{n} = \frac{F}{tl} \quad \text{or} \quad \frac{37.5(10)^3}{2.80} = \frac{5850}{0.375l}$$

or $l = 1.16$ in. To resist crushing, the area of one-half the face of the key is used:

$$\frac{S_y}{n} = \frac{F}{tl/2} \quad \text{or} \quad \frac{65(10)^3}{2.80} = \frac{5850}{0.375l/2}$$

and $l = 1.34$ in. The hub length of a gear is usually greater than the shaft diameter, for stability. If the key, in this example, is made equal in length to the hub, it would therefore have ample strength, since it would probably be $1\frac{7}{16}$ in or longer.

Retaining Rings

- Retaining rings are often used instead of a shoulder to provide axial positioning

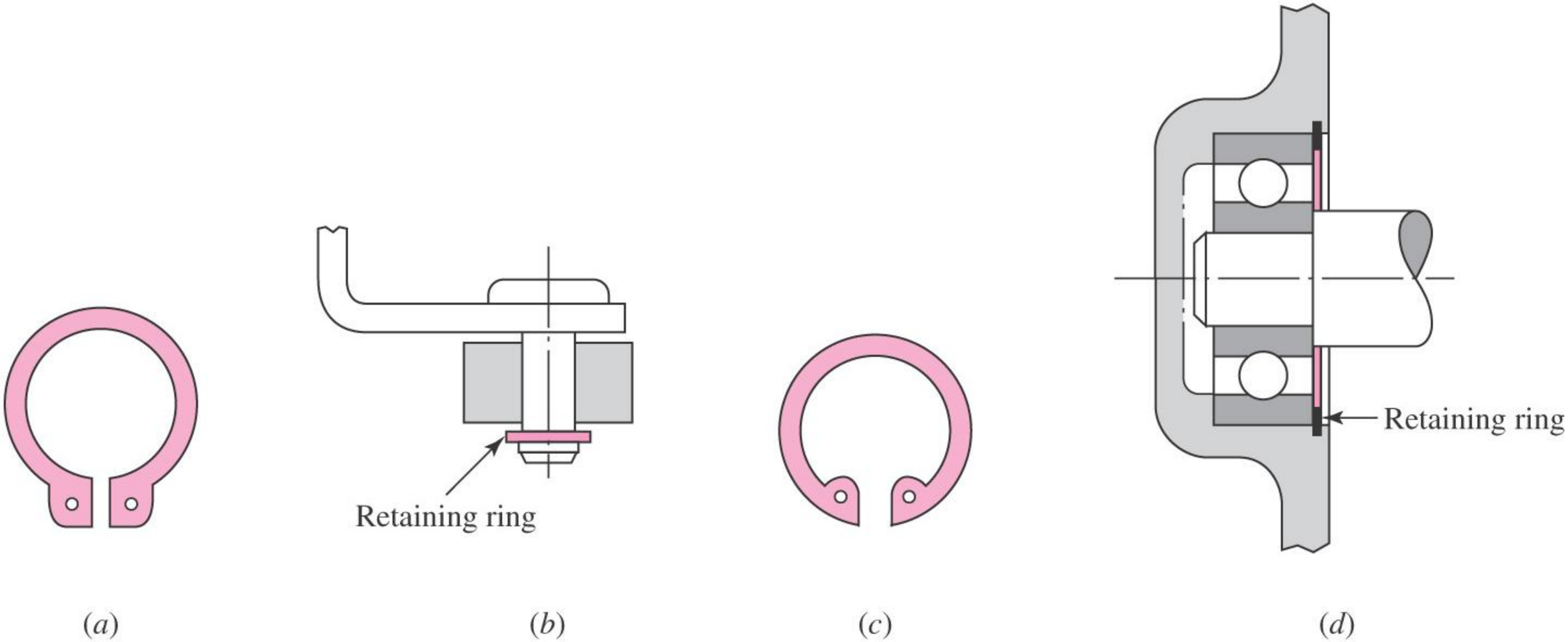


Fig. 7–18

Retaining Rings

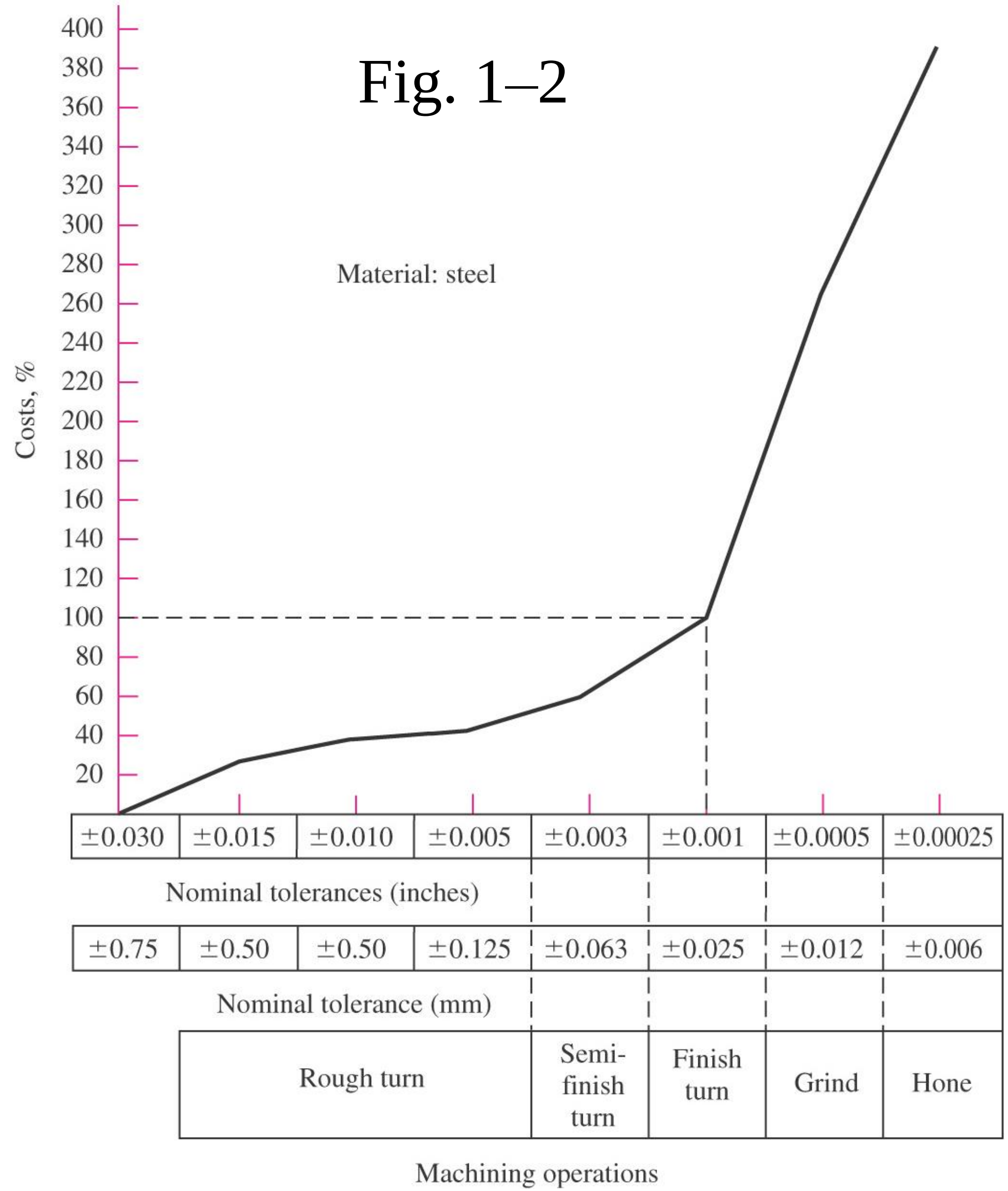
- Retaining ring must seat well in bottom of groove to support axial loads against the sides of the groove.
- This requires sharp radius in bottom of groove.
- Stress concentrations for flat-bottomed grooves are available in Table A–15–16 and A–15–17.
- Typical stress concentration factors are high, around 5 for bending and axial, and 3 for torsion

Limits and Fits

- Shaft diameters need to be sized to "fit" the shaft components (e.g. gears, bearings, etc.)
- Need ease of assembly
- Need minimum slop
- May need to transmit torque through press fit
- Shaft design requires only nominal shaft diameters
- Precise dimensions, including tolerances, are necessary to finalize design

Tolerances

- Close tolerances generally increase cost
  - Require additional processing steps
  - Require additional inspection
  - Require machines with lower production rates



Standards for Limits and Fits

- Two standards for limits and fits in use in United States
  - U.S. Customary (1967)
    - Metric (1978)
- Metric version will be presented, with a set of inch conversions

Nomenclature for Cylindrical Fit

- Upper case letters refer to hole
- Lower case letters refer to shaft
- *Basic size* is the nominal diameter and is same for both parts, *D=d*
- *Tolerance* is the difference between maximum and minimum size
- *Deviation* is the difference between a size and the basic size

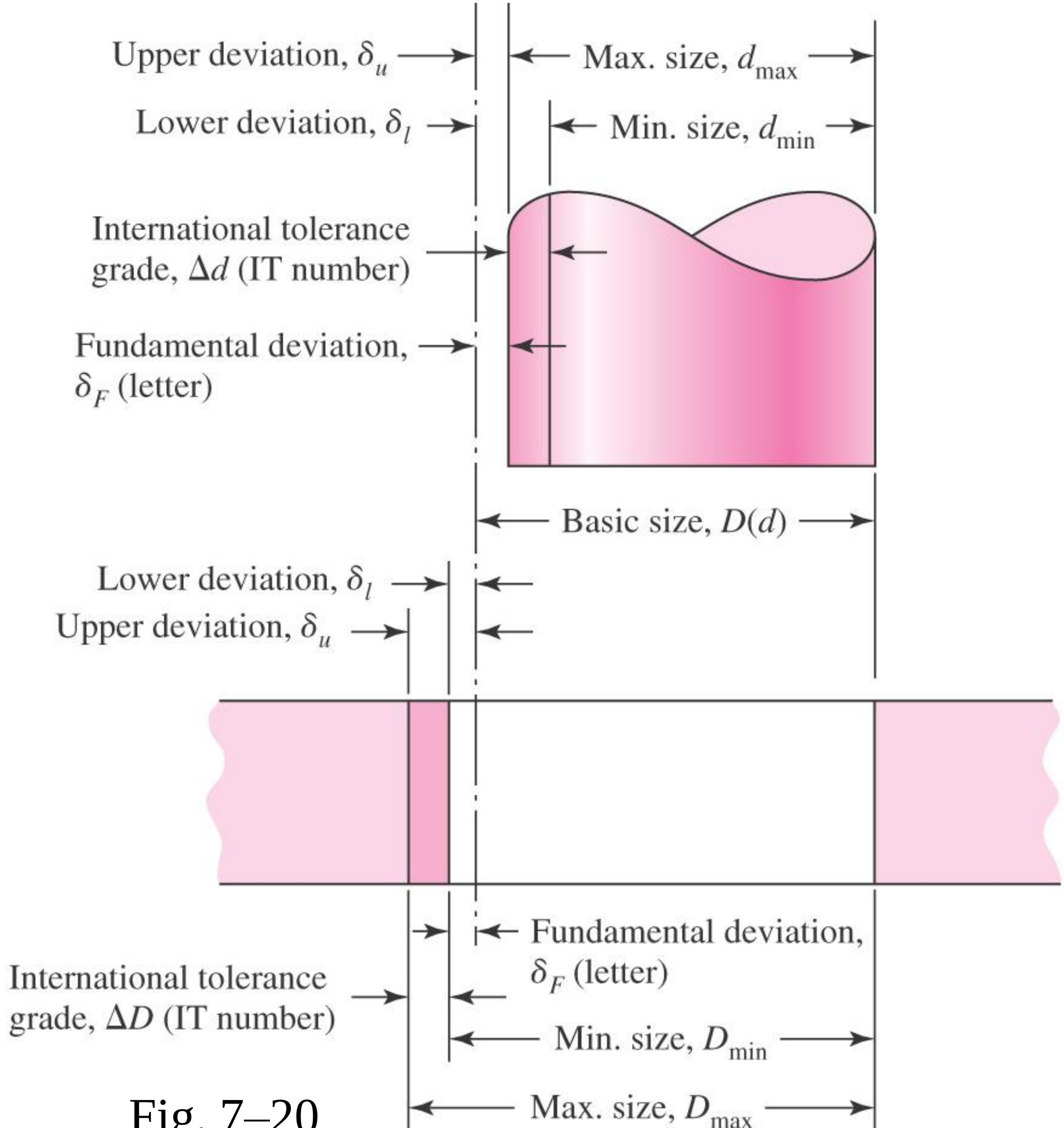


Fig. 7–20

Tolerance Grade Number

- *Tolerance* is the difference between maximum and minimum size
- *International tolerance grade* numbers designate groups of tolerances such that the tolerances for a particular IT number have the same relative level of accuracy but vary depending on the basic size
- IT grades range from IT0 to IT16, but only IT6 to IT11 are generally needed
- Specifications for IT grades are listed in Table A–11 for metric series and A–13 for inch series

Tolerance Grades – Metric Series

Basic Sizes	Tolerance Grades IT6	IT7	IT8	IT9	IT10	IT11
0–3	0.006	0.010	0.014	0.025	0.040	0.060
3–6	0.008	0.012	0.018	0.030	0.048	0.075
6–10	0.009	0.015	0.022	0.036	0.058	0.090
10–18	0.011	0.018	0.027	0.043	0.070	0.110
18–30	0.013	0.021	0.033	0.052	0.084	0.130
30–50	0.016	0.025	0.039	0.062	0.100	0.160
50–80	0.019	0.030	0.046	0.074	0.120	0.190
80–120	0.022	0.035	0.054	0.087	0.140	0.220
120–180	0.025	0.040	0.063	0.100	0.160	0.250
180–250	0.029	0.046	0.072	0.115	0.185	0.290
250–315	0.032	0.052	0.081	0.130	0.210	0.320
315–400	0.036	0.057	0.089	0.140	0.230	0.360

Table A–11

Tolerance Grades – Inch Series

Basic Sizes	Tolerance Grades					
	IT6	IT7	IT8	IT9	IT10	IT11
0–0.12	0.0002	0.0004	0.0006	0.0010	0.0016	0.0024
0.12–0.24	0.0003	0.0005	0.0007	0.0012	0.0019	0.0030
0.24–0.40	0.0004	0.0006	0.0009	0.0014	0.0023	0.0035
0.40–0.72	0.0004	0.0007	0.0011	0.0017	0.0028	0.0043
0.72–1.20	0.0005	0.0008	0.0013	0.0020	0.0033	0.0051
1.20–2.00	0.0006	0.0010	0.0015	0.0024	0.0039	0.0063
2.00–3.20	0.0007	0.0012	0.0018	0.0029	0.0047	0.0075
3.20–4.80	0.0009	0.0014	0.0021	0.0034	0.0055	0.0087
4.80–7.20	0.0010	0.0016	0.0025	0.0039	0.0063	0.0098
7.20–10.00	0.0011	0.0018	0.0028	0.0045	0.0073	0.0114
10.00–12.60	0.0013	0.0020	0.0032	0.0051	0.0083	0.0126
12.60–16.00	0.0014	0.0022	0.0035	0.0055	0.0091	0.0142

Table A–13

Deviations

- *Deviation* is the algebraic difference between a size and the basic size
- *Upper deviation* is the algebraic difference between the maximum limit and the basic size
- *Lower deviation* is the algebraic difference between the minimum limit and the basic size
- *Fundamental deviation* is either the upper or lower deviation that is closer to the basic size
- Letter codes are used to designate a similar level of clearance or interference for different basic sizes

Fundamental Deviation Letter Codes

- Shafts with clearance fits
  - Letter codes c, d, f, g, and h
  - Upper deviation = fundamental deviation
  - Lower deviation = upper deviation – tolerance grade
- Shafts with transition or interference fits
  - Letter codes k, n, p, s, and u
  - Lower deviation = fundamental deviation
  - Upper deviation = lower deviation + tolerance grade
- Hole
  - The standard is a *hole based* standard, so letter code H is always used for the hole
  - Lower deviation = 0 (Therefore $D_{min} = 0$)
  - Upper deviation = tolerance grade
- Fundamental deviations for letter codes are shown in Table A–12 for metric series and A–14 for inch series

Fundamental Deviations – Metric series

Basic Sizes	Upper-Deviation Letter					Lower-Deviation Letter				
	c	d	f	g	h	k	n	p	s	u
0–3	−0.060	−0.020	−0.006	−0.002	0	0	+0.004	+0.006	+0.014	+0.018
3–6	−0.070	−0.030	−0.010	−0.004	0	+0.001	+0.008	+0.012	+0.019	+0.023
6–10	−0.080	−0.040	−0.013	−0.005	0	+0.001	+0.010	+0.015	+0.023	+0.028
10–14	−0.095	−0.050	−0.016	−0.006	0	+0.001	+0.012	+0.018	+0.028	+0.033
14–18	−0.095	−0.050	−0.016	−0.006	0	+0.001	+0.012	+0.018	+0.028	+0.033
18–24	−0.110	−0.065	−0.020	−0.007	0	+0.002	+0.015	+0.022	+0.035	+0.041
24–30	−0.110	−0.065	−0.020	−0.007	0	+0.002	+0.015	+0.022	+0.035	+0.048
30–40	−0.120	−0.080	−0.025	−0.009	0	+0.002	+0.017	+0.026	+0.043	+0.060
40–50	−0.130	−0.080	−0.025	−0.009	0	+0.002	+0.017	+0.026	+0.043	+0.070
50–65	−0.140	−0.100	−0.030	−0.010	0	+0.002	+0.020	+0.032	+0.053	+0.087
65–80	−0.150	−0.100	−0.030	−0.010	0	+0.002	+0.020	+0.032	+0.059	+0.102
80–100	−0.170	−0.120	−0.036	−0.012	0	+0.003	+0.023	+0.037	+0.071	+0.124
100–120	−0.180	−0.120	−0.036	−0.012	0	+0.003	+0.023	+0.037	+0.079	+0.144
120–140	−0.200	−0.145	−0.043	−0.014	0	+0.003	+0.027	+0.043	+0.092	+0.170
140–160	−0.210	−0.145	−0.043	−0.014	0	+0.003	+0.027	+0.043	+0.100	+0.190
160–180	−0.230	−0.145	−0.043	−0.014	0	+0.003	+0.027	+0.043	+0.108	+0.210
180–200	−0.240	−0.170	−0.050	−0.015	0	+0.004	+0.031	+0.050	+0.122	+0.236
200–225	−0.260	−0.170	−0.050	−0.015	0	+0.004	+0.031	+0.050	+0.130	+0.258
225–250	−0.280	−0.170	−0.050	−0.015	0	+0.004	+0.031	+0.050	+0.140	+0.284
250–280	−0.300	−0.190	−0.056	−0.017	0	+0.004	+0.034	+0.056	+0.158	+0.315
280–315	−0.330	−0.190	−0.056	−0.017	0	+0.004	+0.034	+0.056	+0.170	+0.350
315–355	−0.360	−0.210	−0.062	−0.018	0	+0.004	+0.037	+0.062	+0.190	+0.390
355–400	−0.400	−0.210	−0.062	−0.018	0	+0.004	+0.037	+0.062	+0.208	+0.435

Table A–12

Fundamental Deviations – Inch series

Basic Sizes	Upper-Deviation Letter					Lower-Deviation Letter				
	c	d	f	g	h	k	n	p	s	u
0–0.12	−0.0024	−0.0008	−0.0002	−0.0001	0	0	+0.0002	+0.0002	+0.0006	+0.0007
0.12–0.24	−0.0028	−0.0012	−0.0004	−0.0002	0	0	+0.0003	+0.0005	+0.0007	+0.0009
0.24–0.40	−0.0031	−0.0016	−0.0005	−0.0002	0	0	+0.0004	+0.0006	+0.0009	+0.0011
0.40–0.72	−0.0037	−0.0020	−0.0006	−0.0002	0	0	+0.0005	+0.0007	+0.0011	+0.0013
0.72–0.96	−0.0043	−0.0026	−0.0008	−0.0003	0	+0.0001	+0.0006	+0.0009	+0.0014	+0.0016
0.96–1.20	−0.0043	−0.0026	−0.0008	−0.0003	0	+0.0001	+0.0006	+0.0009	+0.0014	+0.0019
1.20–1.60	−0.0047	−0.0031	−0.0010	−0.0004	0	+0.0001	+0.0007	+0.0010	+0.0017	+0.0024
1.60–2.00	−0.0051	−0.0031	−0.0010	−0.0004	0	+0.0001	+0.0007	+0.0010	+0.0017	+0.0028
2.00–2.60	−0.0055	−0.0039	−0.0012	−0.0004	0	+0.0001	+0.0008	+0.0013	+0.0021	+0.0034
2.60–3.20	−0.0059	−0.0039	−0.0012	−0.0004	0	+0.0001	+0.0008	+0.0013	+0.0023	+0.0040
3.20–4.00	−0.0067	−0.0047	−0.0014	−0.0005	0	+0.0001	+0.0009	+0.0015	+0.0028	+0.0049
4.00–4.80	−0.0071	−0.0047	−0.0014	−0.0005	0	+0.0001	+0.0009	+0.0015	+0.0031	+0.0057
4.80–5.60	−0.0079	−0.0057	−0.0017	−0.0006	0	+0.0001	+0.0011	+0.0017	+0.0036	+0.0067
5.60–6.40	−0.0083	−0.0057	−0.0017	−0.0006	0	+0.0001	+0.0011	+0.0017	+0.0039	+0.0075
6.40–7.20	−0.0091	−0.0057	−0.0017	−0.0006	0	+0.0001	+0.0011	+0.0017	+0.0043	+0.0083
7.20–8.00	−0.0094	−0.0067	−0.0020	−0.0006	0	+0.0002	+0.0012	+0.0020	+0.0048	+0.0093
8.00–9.00	−0.0102	−0.0067	−0.0020	−0.0006	0	+0.0002	+0.0012	+0.0020	+0.0051	+0.0102
9.00–10.00	−0.0110	−0.0067	−0.0020	−0.0006	0	+0.0002	+0.0012	+0.0020	+0.0055	+0.0112
10.00–11.20	−0.0118	−0.0075	−0.0022	−0.0007	0	+0.0002	+0.0013	+0.0022	+0.0062	+0.0124
11.20–12.60	−0.0130	−0.0075	−0.0022	−0.0007	0	+0.0002	+0.0013	+0.0022	+0.0067	+0.0130
12.60–14.20	−0.0142	−0.0083	−0.0024	−0.0007	0	+0.0002	+0.0015	+0.0024	+0.0075	+0.0154
14.20–16.00	−0.0157	−0.0083	−0.0024	−0.0007	0	+0.0002	+0.0015	+0.0024	+0.0082	+0.0171

Table A–14

Specification of Fit

- A particular fit is specified by giving the basic size followed by letter code and IT grades for hole and shaft.
- For example, a sliding fit of a nominally 32 mm diameter shaft and hub would be specified as 32H7/g6
- This indicates
  - 32 mm basic size
  - hole with IT grade of 7 (look up tolerance ΔD in Table A–11)
  - shaft with fundamental deviation specified by letter code g (look up fundamental deviation δ_F in Table A–12)
  - shaft with IT grade of 6 (look up tolerance Δd in Table A–11)
- Appropriate letter codes and IT grades for common fits are given in Table 7–9

Description of Preferred Fits (Clearance)

Type of Fit	Description	Symbol
Clearance	*Loose running fit:* for wide commercial tolerances or allowances on external members	H11/c11
	Free running fit: not for use where accuracy is essential, but good for large temperature variations, high running speeds, or heavy journal pressures	H9/d9
	Close running fit: for running on accurate machines and for accurate location at moderate speeds and journal pressures	H8/f7
	Sliding fit: where parts are not intended to run freely, but must move and turn freely and locate accurately	H7/g6
	Locational clearance fit: provides snug fit for location of stationary parts, but can be freely assembled and disassembled	H7/h6

Table 7–9

Description of Preferred Fits (Transition & Interference)

Type of Fit	Description	Symbol
Transition	*Locational transition fit:* for accurate location, a compromise between clearance and interference	H7/k6
	Locational transition fit: for more accurate location where greater interference is permissible	H7/n6
Interference	*Locational interference fit:* for parts requiring rigidity and alignment with prime accuracy of location but without special bore pressure requirements	H7/p6
	Medium drive fit: for ordinary steel parts or shrink fits on light sections, the tightest fit usable with cast iron	H7/s6
	Force fit: suitable for parts that can be highly stressed or for shrink fits where the heavy pressing forces required are impractical	H7/u6

Table 7–9

Procedure to Size for Specified Fit

- Select description of desired fit from Table 7–9.
- Obtain letter codes and IT grades from symbol for desired fit from Table 7–9
- Use Table A–11 (metric) or A–13 (inch) with IT grade numbers to obtain ΔD for hole and Δd for shaft
- Use Table A–12 (metric) or A–14 (inch) with shaft letter code to obtain δ_F for shaft
- For hole

$$D_{\max} = D + \Delta D \qquad D_{\min} = D \qquad (7\text{–}36)$$

- For shafts with clearance fits c, d, f, g, and h

$$d_{\max} = d + \delta_F \qquad d_{\min} = d + \delta_F - \Delta d \qquad (7\text{–}37)$$

- For shafts with interference fits k, n, p, s, and u

$$d_{\min} = d + \delta_F \qquad d_{\max} = d + \delta_F + \Delta d \qquad (7\text{–}38)$$

Example 7-7

Find the shaft and hole dimensions for a loose running fit with a 34-mm basic size.

Solution

From Table 7–9, the ISO symbol is 34H11/c11. From Table A–11, we find that tolerance grade IT11 is 0.160 mm. The symbol 34H11/c11 therefore says that $\Delta D = \Delta d = 0.160$ mm. Using Eq. (7–36) for the hole, we get

$$D_{max} = D + \Delta D = 34 + 0.160 = 34.160 \text{ mm}$$

$$D_{min} = D = 34.000 \text{ mm}$$

The shaft is designated as a 34c11 shaft. From Table A–12, the fundamental deviation is $\delta_F = -0.120$ mm. Using Eq. (7–37), we get for the shaft dimensions

$$d_{max} = d + \delta_F = 34 + (-0.120) = 33.880 \text{ mm}$$

$$d_{min} = d + \delta_F - \Delta d = 34 + (-0.120) - 0.160 = 33.720 \text{ mm}$$

Example 7-8

Find the hole and shaft limits for a medium drive fit using a basic hole size of 2 in.

Solution

The symbol for the fit, from Table 7–8, in inch units is (2 in)H7/s6. For the hole, we use Table A–13 and find the IT7 grade to be $\Delta D = 0.0010$ in. Thus, from Eq. (7–36),

$$D_{\text{max}} = D + \Delta D = 2 + 0.0010 = 2.0010 \text{ in}$$

$$D_{\text{min}} = D = 2.0000 \text{ in}$$

The IT6 tolerance for the shaft is $\Delta d = 0.0006$ in. Also, from Table A–14, the fundamental deviation is $\delta_F = 0.0017$ in. Using Eq. (7–38), we get for the shaft that

$$d_{\text{min}} = d + \delta_F = 2 + 0.0017 = 2.0017 \text{ in}$$

$$d_{\text{max}} = d + \delta_F + \Delta d = 2 + 0.0017 + 0.0006 = 2.0023 \text{ in}$$

Stress in Interference Fits

- Interference fit generates pressure at interface
- Need to ensure stresses are acceptable
- Treat shaft as cylinder with uniform external pressure
- Treat hub as hollow cylinder with uniform internal pressure
- These situations were developed in Ch. 3 and will be adapted here

Stress in Interference Fits

- The pressure at the interface, from Eq. (3–56) converted into terms of diameters,

$$p = \frac{\delta}{\frac{d}{E_o}\left(\frac{d_o^2 + d^2}{d_o^2 - d^2} + \nu_o\right) + \frac{d}{E_i}\left(\frac{d^2 + d_i^2}{d^2 - d_i^2} - \nu_i\right)} \qquad (7\text{–}39)$$

- If both members are of the same material,

$$p = \frac{E\delta}{2d^3}\left[\frac{(d_o^2 - d^2)(d^2 - d_i^2)}{d_o^2 - d_i^2}\right] \qquad (7\text{–}40)$$

- δ is *diametral* interference

$$\delta = d_{\text{shaft}} - d_{\text{hub}} \qquad (7\text{–}41)$$

- Taking into account the tolerances,

$$\delta_{\min} = d_{\min} - D_{\max} \qquad (7\text{–}42)$$

$$\delta_{\max} = d_{\max} - D_{\min} \qquad (7\text{–}43)$$

Stress in Interference Fits

- From Eqs. (3–58) and (3–59), with radii converted to diameters, the tangential stresses at the interface are

$$\sigma_{t,\text{ shaft}} = -p\frac{d^2 + d_i^2}{d^2 - d_i^2} \qquad (7\text{–}44)$$

$$\sigma_{t,\text{ hub}} = p\frac{d_o^2 + d^2}{d_o^2 - d^2} \qquad (7\text{–}45)$$

- The radial stresses at the interface are simply

$$\sigma_{r,\text{ shaft}} = -p \qquad (7\text{–}46)$$

$$\sigma_{r,\text{ hub}} = -p \qquad (7\text{–}47)$$

- The tangential and radial stresses are orthogonal and can be combined using a failure theory

Torque Transmission from Interference Fit

- Estimate the torque that can be transmitted through interference fit by friction analysis at interface

$$F_f = fN = f(pA) = f[p2\pi(d/2)l] = \pi fpld \qquad (7\text{–}48)$$

$$T = F_f d/2 = \pi fpld(d/2)$$

$$T = (\pi/2) fpld^2 \qquad (7\text{–}49)$$

- Use the minimum interference to determine the minimum pressure to find the maximum torque that the joint should be expected to transmit.