CHAPTER FIVE

AC-AC CONVERSION: AC VOLTAGE CONTROLLER

5.1 INTRODUCTION
There are numerous systems in use today that convert the fixed voltage and fixed frequency a.c. supply into variable voltage or and variable frequency supply using power semiconductor devices. The simplest forms of ac-to-ac converters are the a.c. voltage controllers that convert fixed voltage fixed frequency into variable voltage fixed frequency. These voltage controllers are also called a.c. choppers or a.c. voltage regulators. Some of the applications are motor drive systems; electric furnaces heat control, light dimmers, HVAC systems, welding and other industrial applications. This chapter discusses the single-phase and three-phase a.c. voltage controller (a.c. choppers) and their performance with resistive and resistive-inductive loads.

5.2 SINGLE-PHASE AC VOLTAGE CONTROLLER
A single-phase a.c. voltage controller is shown in Fig.5.1. In this controller two thyristors (SCRs) are connected in inverse-parallel (back-to-back) to perform the function of an electronic switch suitable for use with a.c supply. If suitable gating pulses are applied to the thyristors while their respective anode voltage is positive, current conduction is initiated. The conduction angle depends on the triggering-angle, which is measured form anode voltage zero.
Two different modes of switching operation are mainly used:

(a) If each switch triggered at some non-zero point on its respective anode voltage cycle, the voltage waveform is described as phase-angle control.
(b) If triggering is used to permit complete cycles of load current followed by complete cycles of extinction, the load voltage and current waveform is variously described as burst firing, zero voltage switching, cycle selection, on-off control or integral-cycle switching.

Also in phase-angle control mode of operation, two different modes may be obtained:

(c) If $\alpha_1 = \alpha_2 = \alpha$, i.e. the two thyristors are triggered at same angle value, then the triggering mode is called symmetrical angle triggering.
(d) If triggering is used with two different values of angles $\alpha_1 \neq \alpha_2$, then the triggering mode is called asymmetrical angle triggering.

### 5.2.1 AC Voltage Controller Working with Resistive Load and Symmetrical Phase-Angle Triggering

With resistive load single gating pulse of magnitude 1-3 V is usually sufficient to switch on an SCR. Waveforms for sinusoidal supply voltage with an arbitrary triggering angle $\alpha = 60^\circ$ is given in Fig.5.2. The gate current pulse $i_g$ in both the thyristors should be positive, since the thyristor does not work with negative gate current at all.
Analytical properties of the output voltage waveform

The voltage waveforms are defined in terms of the maximum value of the supply phase voltage $V_m$ by

- The supply voltage $v_s$

  $$v_s(\omega t) = V_m \sin \omega t$$  \hspace{1cm} (5.1)

- The load voltage $v_L$

  $$v_L = V_m \sin \omega t \begin{cases} \pi, 2\pi, \ldots \\ \alpha, \pi + \alpha, \ldots \end{cases}$$

  $$v_L = 0 \quad \text{elsewhere}$$  \hspace{1cm} (5.2)

- The thyristor voltage $v_T$

  $$v_T = V_m \sin \omega t \begin{cases} \alpha, \pi + \alpha, \ldots \\ 0, \pi, \ldots \end{cases}$$  \hspace{1cm} (5.3)
5.2.2 The \textit{rms} Values of the Load Voltage and Current

Any function \( v_L(\omega t) \) that is periodic in \( 2\pi \) radians has a root mean square (\textit{rms}) or effective value defined by:

\[
V_L = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} v_L^2(\omega t) d\omega t
\]  
(5.4)

The function \( v_L(\omega t) \) is defined by Eq.(5.2) for the circuit of Fig.5.1. Substituting Eq.(5.2) into Eq.(5.4) gives:

\[
V_L = \frac{1}{\sqrt{2\pi}} \int_0^{\alpha} (V_m \sin \omega t)^2 d\omega t + \frac{1}{\sqrt{2\pi}} \int_{\alpha+\pi}^{2\pi} (V_m \sin \omega t)^2 d\omega t
\]

But \( \frac{1}{\sqrt{2\pi}} \int_0^{\pi} (V_m \sin \omega t)^2 d\omega t = \frac{1}{\sqrt{2\pi}} \int_{\alpha+\pi}^{2\pi} (V_m \sin \omega t)^2 d\omega t \)

Hence, it can be written as

\[
V_L = \sqrt{\frac{1}{\pi} \int_0^{\pi} (V_m \sin \omega t)^2 d\omega t} = \sqrt{\frac{V_m^2}{\pi} \int_0^{\pi} \frac{1}{2} (1 - \cos 2\omega t) d\omega t}
\]

\[
V_L = \frac{V_m}{\sqrt{2}} \sqrt{\frac{1}{\pi} \left\{ (\pi - \alpha) + \frac{\sin 2\alpha}{2} \right\}}
\]  
(5.5)

The \textit{rms} value of the sinusoidal supply voltage is given by the standard relationship:

\[
V_s = \frac{V_m}{\sqrt{2}}
\]

The \textit{rms} load voltage \( V_L \) given in Eq.(5.5) can, therefore, be re-written as

\[
V_L = V_s \sqrt{\frac{1}{\pi} \left\{ (\pi - \alpha) + \frac{\sin 2\alpha}{2} \right\}}
\]  
(5.6)

With resistive load the instantaneous current is given by:

\[
i_L = \frac{v_L}{R} = \frac{V_m}{R} \sin \omega t \left\{ \frac{\pi}{\alpha}, \frac{2\pi}{\alpha}, ... \right\}
\]  
(5.7)
The \textit{rms} load current $I_L$ can therefore be written directly from:

$$I_L = \frac{V_m}{\sqrt{2}R} \sqrt{\frac{1}{\pi} \left( \pi - \alpha \right) + \frac{\sin 2\alpha}{2}}$$ \hspace{1cm} (5.8)

In terms of the \textit{rms} supply voltage $V_s$ the \textit{rms} current is given by:

$$I_L = \frac{V_s}{R} \sqrt{\frac{1}{\pi} \left( \pi - \alpha \right) + \frac{\sin 2\alpha}{2}}$$ \hspace{1cm} (5.9)

\textbf{Power and power factor}

The average power delivered to the load:

For resistive load, the average load power $P_L$ is

$$P_L = I^2R \quad \text{or} \quad P_L = \frac{V^2}{R}$$ \hspace{1cm} (5.10)

$$P_L = \frac{V_s^2}{R} \left[ \frac{1}{\pi} \left( \pi - \alpha \right) + \frac{\sin 2\alpha}{2} \right]$$ \hspace{1cm} (5.11)

The apparent power taken from the supply

$$P_s = V_s \cdot I_L$$

$$P_s = \frac{V_s^2}{R} \sqrt{\frac{1}{\pi} \left( \pi - \alpha \right) + \frac{\sin 2\alpha}{2}}$$ \hspace{1cm} (5.12)

Input power factor

$$PF = \frac{\text{Average power}}{\text{Apparent power}}$$

$$PF = \frac{V_L \cdot I_L}{V_s \cdot I_L} = \frac{V_L}{V_s} = \frac{1}{\pi} \left( \pi - \alpha \right) + \frac{\sin 2\alpha}{2}$$ \hspace{1cm} (5.13)
5.3 HARMONICS ANALYSIS OF THE LOAD VOLTAGE WAVEFORM OF AC VOLTAGE CONTROLLER

The load voltage \( v_L \) waveform shown in Fig.5.3 is a nonsinusoidal waveform that contains harmonic components.

![Load voltage waveform](image)

Fig.5.3 Load voltage waveform of a single-phase a.c. voltage controller with resistive load.

In terms of harmonics, this waveform can be expressed in Fourier series (see Appendix-A) as

\[
v_L(\omega t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t
\]

(5.14)

\[
v_L(\omega t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \sin(n\omega t + \psi_n)
\]

(5.15)

where, the Fourier coefficients are,

\[
a_0 = \frac{1}{2\pi} \int_{0}^{2\pi} V_L(\omega t) \, d\omega t
\]

(5.16)

\[
a_n = \frac{1}{\pi} \int_{0}^{2\pi} v_L(\omega t) \cos n\omega t \, d\omega t
\]

(5.17)

\[
b_n = \frac{1}{\pi} \int_{0}^{2\pi} V_L(\omega t) \sin n\omega t \, d\omega t
\]

(5.18)

\( n = \) the \( n^{th} \) order harmonics (\( n = 1, 2, 3, \ldots \))

\( c_n = \) the amplitude of the \( n^{th} \) order harmonics

\[
c_n = \sqrt{a_n^2 + b_n^2}
\]

(5.19)
\( \psi_n = \text{phase angle of the } n^{\text{th}} \text{ harmonic component given as,} \)

\[
\psi_n = \tan^{-1} \frac{a_n}{b_n} \tag{5.20}
\]

For the fundamental component (\( n = 1 \)):

\[
a_1 = \frac{1}{\pi} \int_0^{2\pi} v_L(\omega) \cos \omega t \, dt \tag{5.21}
\]

\[
b_1 = \frac{1}{\pi} \int_0^{2\pi} V_L(\omega) \sin \omega t \, dt \tag{5.22}
\]

The amplitude of the fundamental component is

\[
c_1 = \sqrt{a_1^2 + b_1^2} \tag{5.23}
\]

The phase angle of the fundamental component is given as,

\[
\psi_1 = \tan^{-1} \frac{a_1}{b_1} \tag{5.24}
\]

For the fundamental component of the load voltage, referring to Fig.5.5,

\[
a_1 = \frac{1}{\pi} \int_0^{2\pi} V_m \sin \omega t \cos \omega t \, dt
\]

\[
= \frac{1}{\pi} \int_0^{\pi} V_m \sin \omega t \cos \omega t \, dt + \frac{1}{\pi} \int_{\alpha+\pi}^{2\pi} V_m \sin \omega t \cos \omega t \, dt
\]

\[
= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} V_m \sin 2\omega t \, dt \quad \text{(since : } \sin 2\omega t = 2 \sin \omega t \cos \omega t)\]

\[
= \frac{1}{\pi} \int_0^{\pi} V_m \sin 2\omega t \, dt = \frac{V_m}{2\pi} \left[ -\cos 2\pi - \cos 2\alpha \right]
\]

\[
\therefore a_1 = \frac{V_m}{2\pi} (\cos 2\alpha - 1) \tag{5.25}
\]
Similarly,

\[ b_1 = \frac{1}{\pi} \int_0^{2\pi} V_m \sin \omega t \sin \omega t \, d\omega t \]

\[ = \frac{1}{\pi} \int_0^{\pi} V_m \sin \omega t \sin \omega t \, d\omega t + \frac{1}{\pi} \int_{\pi}^{2\pi} V_m \sin \omega t \sin \omega t \, d\omega t \]

\[ = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} V_m (1 - \cos 2\omega t) \, d\omega t \]

since \((\sin \omega t)^2 = \frac{1}{2}(1 - \cos 2\omega t)\)

\[ \therefore b_1 = \frac{V_m}{2\pi} (2\pi - 2\alpha + \sin 2\alpha) \] (5.26)

\[ c_1 = \sqrt{\frac{V_m^2}{2\pi} (\cos 2\alpha - 1)^2 + \left[ \frac{V_m}{2\pi} (2\pi - 2\alpha + \sin 2\alpha) \right]^2} \] (5.27)

\[ \psi_1 = \tan^{-1} \frac{(\cos 2\alpha - 1)}{(2\pi - 2\alpha + \sin 2\alpha)} \] (5.28)

For the \(n^{th}\) Fourier harmonic component, using Eq.(5.17) and Eq.(5.18), we get,

\[ a_n = \frac{V_m}{2\pi} \left[ \frac{1}{n + 1} \{1 + (-1)^{n+1}\} \{\cos(n + 1)\alpha - 1\} \right. \]

\[ - \frac{1}{n - 1} \{1 + (-1)^{n-1}\} \{\cos(n - 1)\alpha - 1\} \] (5.29)

\[ b_n = \frac{V_m}{2\pi} \left[ \frac{1}{n + 1} \{1 + (-1)^{n+1}\} \{\sin(n + 1)\alpha - 1\} \right. \]

\[ - \frac{1}{n - 1} \{1 + (-1)^{n-1}\} \{\sin(n - 1)\alpha - 1\} \] (5.30)

- For even order harmonic terms \((n = 2,4,6,\ldots)\) the values of \((-1)^{n+1}\) and \((-1)^{n-1}\) are \((-1)\), hence \(a_n = 0\) and \(b_n = 0\) (no even harmonics).
- For odd harmonic terms \((n = 3,5,7,\ldots)\) the values of \((-1)^{n+1}\) and \((-1)^{n-1}\) are unity. Hence \(a_n \neq 0\) and \(b_n \neq 0\).

Hence, this will modify the above equations to:
\[ a_n = \frac{V_m}{2\pi} \left[ \frac{2}{n+1} \{ \cos(n+1) \alpha - 1 \} - \frac{2}{n-1} \{ \cos(n-1) \alpha - 1 \} \right] \] (5.31)

\[ b_n = \frac{V_m}{2\pi} \left[ \frac{2}{n+1} \{ \sin(n+1) \alpha - 1 \} - \frac{2}{n-1} \{ \sin(n-1) \alpha - 1 \} \right] \] (5.32)

Therefore, only odd harmonics: 3rd, 5th, 7th, 9th, 11th…etc are exist in the load voltage waveform of Fig. 5.3. The fundamental component \( v_1 = c_1 \sin \omega t \) has same frequency as that of the load voltage waveform (i.e. 50 Hz in our case) whereas the third order harmonics \( v_3 = c_3 \sin 3\omega t \) (i.e. has frequency 3x50 =150 Hz). The variation of these harmonics with the triggering angle \( \alpha \) is shown in Fig. 5.4. Also, Fig. 5.5 shows the time variation of the fundamental, third and fifth order harmonic components of the load voltage waveform for clarity.

![Graph showing variation of harmonic amplitudes with the triggering angle \( \alpha \) in terms of the fundamental component.](image)
5.4 CALCULATION OF POWER DISSIPATION IN TERMS OF HARMONICS

In general the average value of the power is calculated as given in Eq.(5.13). However, the power calculation in terms of harmonic components is

\[ P_L = R \left( I_1^2 + I_3^2 + \ldots \ldots \right) = \frac{1}{R} \left( V_{L1}^2 + V_{L3}^2 + V_{L5}^2 + \ldots \ldots \right) \]

Where \( I_1 \) is the \textit{rms} value of the fundamental current,

\[ I_1 = \frac{V_1}{\sqrt{2}} \times \frac{1}{R} \]

\[ \cos \psi_1 = \frac{b_1}{c_1} \]

**Power factor in systems with sinusoidal voltage (at supply) but non-sinusoidal current**

In general, the power factor is defined as

\[ PF = \frac{\text{Average power}}{\text{Apparent power}} = \frac{P_L}{V_s I_s} \]
where 

\[ V_s = \text{rms value of the supply voltage} \]
\[ I_s = \text{rms value of the supply current} \]

\( i(\omega t) \) is periodic in 2\( \pi \) but is non-sinusoidal.

Average power is obtained by combining in-phase voltage and current components of the same frequency.

\[ P_L = V_s I_1 \cos \psi \]

\[ PF = \frac{V_s I_1 \cos \psi}{V_s I_s} = \frac{I_1}{I_s} \cos \psi \]

\( PF = \text{Distortion factor} \times \text{Displacement factor} \)

where \( \text{Distortion factor} = \frac{I_1}{I_s} \) \hspace{1cm} (5.37) \]

\( \text{Displacement factor} = \cos \psi_1 \) \hspace{1cm} (5.38) \]

- Distortion Factor = 1 for sinusoidal operation.
- Displacement factor is a measure of displacement between \( v(\omega t) \) and \( i(\omega t) \).
- Displacement Factor = 1 for sinusoidal resistive operation.

**Example 5.1**

In a single-phase resistive circuit in which the load voltage is controlled by symmetrical phase angle triggering of a pair of inverse-parallel connected thyristors as shown in Fig.5.1. The output voltage waveform is as shown in Fig.5.2. If the supply voltage is \( v_s = 200 \sin \omega t \), \( \alpha = 60^\circ \) and \( R = 50 \, \Omega \), calculate:

(a) The \( \text{rms} \) value of the fundamental component of the load current,
(b) The value of the displacement angle \( \psi_1 \) between the supply voltage and the fundamental component of the load current,
(c) The \( \text{rms} \) value of the load voltage and the power factor of the circuit.
(d) The displacement factor and the distortion factor.
Solution

(a) The peak value of the fundamental component of the load voltage is found as,

\[ c_1 = \frac{V_m}{2\pi} \sqrt{(\cos 2\alpha - 1)^2 + [2(\pi - \alpha) + \sin 2\alpha]^2} \]

\[ V_m = 200 \text{ V}, \alpha = 60^\circ = \pi/3 \]

\[ c_1 = \frac{200}{2\pi} \sqrt{(\cos 2 \times 60^\circ - 1)^2 + [2\left(\frac{\pi}{3}\right) + \sin 2 \times 60^\circ]^2} \]

\[ c_1 = 167.8 \text{ V} \]

The \textit{rms} value of the fundamental component of load voltage \( V_{L1} \) is calculated as,

\[ V_{L1} = \frac{c_1}{\sqrt{2}} = \frac{167.8}{\sqrt{2}} = 118.7 \text{ V} \]

The fundamental component of the load current is,

\[ I_{L1} = \frac{V_{L1}}{R} = \frac{118.7}{50} = 2.37 \text{ A} \]

(b) The displacement angle \( \psi_1 \)

\[ \psi_1 = \tan^{-1} \frac{a_1}{b_1} = \tan^{-1} \frac{\cos 2\alpha - 1}{2(\pi - \alpha) + \sin 2\alpha} \]

\[ \psi_1 = \tan^{-1} \frac{\cos 2 \times 60^\circ - 1}{2\left(\frac{\pi}{3}\right) + \sin 2 \times 60^\circ} = -16.5^\circ \]

(c) The \textit{rms} value of the load voltage is given by,

\[ V_L = V_m \sqrt{\frac{1}{2\pi} \left\{ (\pi - \alpha) + \frac{\sin 2\alpha}{2} \right\}} \]

\[ V_L = 200 \sqrt{\frac{1}{2\pi} \left\{ \left(\pi - \frac{\pi}{3}\right) + \frac{\sin 120^\circ}{2} \right\}} = 126.85 \text{ V} \]
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\[
P F = \frac{V_L \cdot I_L}{V_s \cdot I_L} = \frac{V_L}{V_s} = \sqrt{\frac{1}{\pi} \{ (\pi - \alpha) + \frac{1}{2} \sin 2\alpha \}}
\]

\[
P F = \frac{V_L \cdot I_L}{V_s \cdot I_L} = \frac{V_L}{V_s} = \frac{126.85}{200/\sqrt{2}} = 0.897
\]

or \( PF = \sqrt{\frac{1}{\pi} \{ (\pi - \frac{\pi}{3}) + \frac{1}{2} \sin 120^\circ \} } = 0.897 \)

(d) The displacement factor :

\[
\cos \psi_1 = \cos(-16.5) = \cos 16.5^\circ = 0.9588
\]

The distortion factor :

\[
DF = \frac{I_{L1}}{I_L} = \frac{2.37}{\frac{126.85}{50}} = 0.934
\]

\( PF \) can also be calculated from :

\[PF = \text{displacement factor} \times \text{distortion factor}\]

\[PF = 0.9588 \times 0.934 = 0.897\]

**Example 5.2**

The a.c. controller in Example 5.1, is used to control the power supplied to a 25 Ω resistor from a single-phase, 230 V, 50 Hz supply. When the triggering angle is 30° the measured \( rms \) load current and voltage are 9 A and 216 V respectively. Determine the ideal theoretical values for the \( rms \) current and voltage and suggest reasons for any difference.

**Solution**

The theoretical value of the \( rms \) load voltage is obtained using Eq.(5.6),

For \( \alpha = 30^\circ \) or \( \frac{\pi}{6} \) rad

\[
V_L = V_s \sqrt{\frac{1}{\pi} \left( \pi - \alpha + \frac{1}{2} \sin 2\alpha \right)}
\]
The measured values are:

\[ \begin{align*}
V_{L_{\text{rms}}} &= 216 \text{ V} \\
I_{L_{\text{rms}}} &= 9.06 \text{ A}
\end{align*} \]

The multimeter or the moving coil instrument measures only the 50 Hz component which is in \( V_{L_{\text{rms}}} \) while the theoretical value of \( \text{rms} \) of the waveform includes all the \( \text{rms} \) values of the harmonic components associated with the waveform, therefore it is 226.6 V.

Example 5.3

An a.c. voltage controller shown in Fig. 5.1 is used to provide power to a single-phase \( R \)-load from an ideal supply \( v_s = V_m \sin \omega t \) by symmetrical phase-angle triggering of a pair of inverse-parallel connected SCRs in the supply lines. Sketch, approximately to scale, the load voltage waveform for firing-angle for \( \alpha = 90^\circ \) and also sketch the corresponding fundamental component of the load current. Derive, from first principle, an expression for the \( \text{rms} \) load current \( I_L \) at any arbitrary angle \( \alpha \), in terms of \( V_m \), \( R \) and \( \alpha \). Use \( I_L \), calculate the per-unit average power dissipated in the load at \( \alpha = 90^\circ \).

Solution

The waveforms of the load voltage and current are shown in Fig. 5.6.

\[ \begin{align*}
V_L &= 230 \sqrt{\frac{1}{\pi} \left[ \left( \pi - \frac{\pi}{6} \right) + \frac{1}{2} \sin 120^\circ \right]} \\
&= 230 \times \sqrt{0.985} = 226.6 \text{ V} \\
I_L &= \frac{V_L}{20} = \frac{226.6}{25} = 9.06 \text{ A}
\]
From Eq. (5.5) the \textit{rms} value of the load voltage is
\[
V_L = V_m \sqrt{\frac{1}{2\pi} \left[ (\pi - \alpha) + \left( \frac{\sin 2\alpha}{2} \right) \right]}
\]

The load current is given by
\[
i_L(\omega t) = \frac{V_m}{R} \sin \omega t
\]

Therefore, the \textit{rms} value of the load current is
\[
I_L = \frac{V_L}{R} = \frac{V_m}{\sqrt{2}R} \sqrt{\frac{1}{\pi} \left[ (\pi - \alpha) + \frac{\sin 2\alpha}{2} \right]}
\]

\[
P = I_L^2 R = \frac{V_L^2}{R} \left[ \frac{1}{\pi} \left[ (\pi - \alpha) + \frac{\sin 2\alpha}{2} \right] \right]
\]

Let \( \frac{V_L^2}{R} = 1.0 \text{ p.u.} \)

For \( \alpha = 80^\circ = 1.3 \text{ rad} \)

\[
P = 1.0 \left[ \frac{1}{\pi} \left[ (\pi - \frac{\pi}{2}) + \frac{\sin 180^\circ}{2} \right] \right] = 0.707 \text{ p.u.}
\]

\begin{example}
\textbf{Example 5.4}

A single-phase resistive heating load is to be controlled from a single-phase, 50 Hz, a.c. supply by means of an inverse parallel pair of thyristors. What will be the firing angle of the thyristors when a load has 67\% of its maximum value?

\textbf{Solution}

From Eqs. (5.6) and (5.9), the \textit{rms} values of the load voltage and load current are
\[
V_L = \frac{V_m}{\sqrt{2}} \sqrt{\frac{1}{\pi} \left( \pi - \alpha + \frac{1}{2} \sin 2\alpha \right)}
\]
\[
I_L = \frac{V_L}{R} = \frac{V_m}{\sqrt{2}R} \sqrt{\frac{1}{\pi} \left( \pi - \alpha + \frac{1}{2} \sin 2\alpha \right)}
\]
\[ P = I_L^2 R = \frac{V_m^2}{2\pi R} \left[ \pi - \alpha + \frac{1}{2} \sin 2\alpha \right] \]

\[ P_{\text{max}} = P \bigg|_{\alpha=0} = \frac{V_m^2}{2R} \]

When \( P_{\text{load}} = 0.67 \)

\[ 0.67 \frac{V_m^2}{2R} = \frac{V_m^2}{2\pi R} \left[ \pi - \alpha + \frac{1}{2} \sin 2\alpha \right] \]

\[ 0.67 = \frac{1}{\pi} \left( \pi - \alpha + \frac{1}{2} \sin 2\alpha \right) \]

\[ 0.67\pi - \pi = -\alpha + \frac{1}{2} \sin 2\alpha \]

\[ \therefore \alpha - \frac{1}{2} \sin 2\alpha = 0.33\pi = 1.036 \]

Solve numerically for \( \alpha \):

<table>
<thead>
<tr>
<th>( \alpha ) (degree)</th>
<th>( \alpha ) (rad..)</th>
<th>( RHS = \alpha - \frac{1}{2} \sin 2\alpha )</th>
<th>LHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>60°</td>
<td>1.046</td>
<td>0.618</td>
<td>0.613</td>
</tr>
<tr>
<td>70°</td>
<td>1.22</td>
<td>0.899</td>
<td>0.898</td>
</tr>
<tr>
<td>80°</td>
<td>1.40</td>
<td>1.229</td>
<td>1.228</td>
</tr>
<tr>
<td>75°</td>
<td>1.308</td>
<td>1.058</td>
<td>1.058</td>
</tr>
</tbody>
</table>

\[ \therefore \alpha \approx 75° \quad \text{LHS=RHS} \]

**Example 5.5**

A single-phase full-wave a.c. voltage controller consists of two thyristors connected in inverse-parallel as shown in Fig.5.7. The supply voltage is \( V_s = 230 \text{ V(rms)} \) at 50 Hz. The controller is used as a light dimmer to control the light intensity of a group of lamps with total resistance of 10 \( \Omega \). The triggering mode of the thyristors is symmetrical with \( \alpha = 60° \).

(a) Sketch compatible waveforms for the supply, load and thyristor voltages.

(b) Calculate the \textit{rms} value of the load voltage.
(c) Determine the input power factor of the circuit.
(d) If one thyristor is replaced by a diode, what will be the \textit{rms} value of the load voltage?

![Fig.5.7 The circuit.](image)

**Solution**

(a) The waveforms for the supply, load and thyristor voltages are shown in Fig.5.2,

(b) The \textit{rms} value of the load voltage is given by Eq. (5.6) as,

\[
V_L = V_s \sqrt{\frac{1}{\pi} \left( \frac{\pi - \alpha}{\alpha} \right) + \frac{1}{2} \sin 2\alpha}
\]

\[
= 230 \sqrt{\frac{1}{\pi} \left( \frac{\pi}{3} \right) + \frac{1}{2} \sin 2 \times 60^\circ}
\]

Hence \( V_L = 206.3 \text{ V} \)

(c) The input power factor of the circuit,

\[
PF = \frac{V_L \cdot I_L}{V_s \cdot I_L} = \frac{V_L}{V_s} = \frac{206.3}{230} = 0.896
\]

(d) If one thyristor is replaced by a diode, the circuit will be as shown in Fig.5.8(a), and the load voltage waveform will is as depicted in Fig.5.8(b). Hence, the \textit{rms} value of the output voltage will be given by

\[
V_L = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} \left( V_m \sin \omega t \right)^2 d\omega t}
\]
and the $rms$ value of the output voltage is, therefore

\[
V_L = \sqrt{\frac{V_m^2}{2\pi} \int_{\alpha}^{2\pi} \frac{1}{2} (1 - \cos 2\omega t) \, d\omega t} = \frac{V_m}{2} \sqrt{\frac{1}{\pi} \left( \frac{2\pi - \alpha}{2} + \frac{\sin 2\alpha}{2} \right)}
\]

\[
= \frac{\sqrt{2} \times 230}{2} \sqrt{\frac{1}{\pi} \left[ \frac{2\pi - \pi}{3} + \frac{\sin 2 \times 60^\circ}{2} \right]} = = 218.46 \text{ V}
\]

Thus, the $rms$ value of the output voltage is increased.

**Example 5. 6**

The circuit arrangement shown in Fig.5.9 is used to vary the current in resistor $R_B$. The triac is triggered so as to produce a periodic load voltage with identical positive and negative alternations.
(a) Sketch the waveform of current in resistor \(R_B\) if the triac firing angle is \(\alpha = 90^\circ\). Assume that \(v = V_m \sin \omega t\).

(b) Show that the Fourier coefficients for the fundamental component of the current are given by:

\[
a_1 = \frac{V_m}{2\pi} \left(\cos 2\alpha - 1\right) \frac{R_A}{R_B(R_A + R_B)} \\
b_1 = \frac{V_m}{2\pi} \left(\sin 2\alpha + 2(\pi - \alpha)\right) \frac{R_A}{R_B(R_A + R_B)} + \frac{V_m}{(R_A + R_B)}
\]

(c) Calculate the value of the fundamental current \(I_{B1}\) in resistor \(R_B\) when \(R_A = R_B = 10 \Omega\), \(V_m = 100\) V and \(\alpha = \pi/2\). Also calculate the phase-angle of the fundamental current \(I_B\) with respect to the supply voltage. Does this value of phase-angle suggest a power factor other than unity? If so, what happens to the fundamental reactive power?

Solution

(a) Current through resistance \(R_B\), \(i_B\) is shown in Fig.5.10.

(b) When triac is OFF for the intervals \(0 \leq \omega t \leq 90^\circ\) and \((\pi < \omega t \leq \pi + 90^\circ)\), the current \(i_{1B}\) through resistance \(R_B\) is:

\[
i_{1B} = \frac{V_m \sin \omega t}{R_A + R_B}
\]

When triac is ON for the positive and negative half-cycles of the supply for the interval \((90^\circ < \omega t \leq \pi)\) and \((90^\circ + \pi < \omega t \leq 2\pi)\) respectively, the current through resistance \(R_B\), \(i_{2B}\) is:
The Fourier coefficients of the fundamental component of the current $i_B$ are,

$$a_1 = \frac{2}{\pi} \left[ \int_0^\alpha \frac{V_m \sin \omega t}{R_A + R_B} \cos \omega t \, d\omega t + \int_\alpha^\pi \frac{V_m \sin \omega t}{R_B} \cos \omega t \, d\omega t \right]$$

$$= \frac{2V_m}{\pi} \left[ \frac{1}{R_A + R_B} \int_0^\alpha \frac{1}{2} \sin 2\omega t \, d\omega t + \int_\alpha^\pi \frac{1}{2R_B} \sin 2\omega t \, d\omega t \right]$$

$$= \frac{2V_m}{\pi} \left[ \frac{1}{R_A + R_B} \left( -\frac{1}{2} \cos 2\omega t \right) + \frac{1}{R_B} \left( -\frac{1}{2} + \frac{1}{2} \cos 2\alpha \right) \right]$$

$$= \frac{2V_m}{\pi} (\cos 2\alpha - 1) \left[ \frac{1}{R_A + R_B} + \frac{1}{R_B} \right]$$

$$= \frac{2V_m}{\pi} (\cos 2\alpha - 1) \left[ \frac{R_A + R_B - R_B}{R_B(R_A + R_B)} \right]$$

$$= \frac{V_m}{\pi} (\cos 2\alpha - 1) \left[ \frac{R_A}{R_B(R_A + R_B)} \right]$$

$$b_1 = \frac{2}{\pi} \left[ \int_0^\alpha \frac{V_m \sin \omega t}{R_A + R_B} \sin \omega t \, d\omega t + \int_\alpha^\pi \frac{V_m \sin \omega t}{R_B} \sin \omega t \, d\omega t \right]$$

$$= \frac{2V_m}{\pi} \left[ \frac{1}{R_A + R_B} \int_0^\alpha \sin \omega t \, d\omega t + \frac{1}{R_B} \int_\alpha^\pi \sin^2 \omega t \, d\omega t \right]$$

$$= \frac{2V_m}{\pi} \left[ \frac{1}{2(R_A + R_B)} \int_0^\alpha (1 - \cos 2\omega t) \, d\omega t \right]$$

$$+ \frac{1}{2R_B} \int_\alpha^\pi (1 - \cos 2\omega t) \, d\omega t$$

$$= \frac{V_m}{\pi} \left[ \frac{1}{R_A + R_B} \left[ \omega t - \frac{1}{2} \sin 2\omega t \right] + \frac{1}{R_B} \left[ \left( \pi - \alpha + \frac{1}{2} \sin 2\alpha \right) \right] \right]$$
= \frac{V_m}{\pi} \left[ \frac{1}{R_A + R_B} \left[ \alpha - \frac{1}{2} \sin 2\alpha \right] + \frac{1}{R_B} \left[ \left( \pi - \alpha + \frac{1}{2} \sin 2\alpha \right) \right] \right]

= \frac{V_m}{\pi( R_A + R_B )} \left( \alpha - \frac{1}{2} \sin 2\alpha \right) + \frac{V_m}{R_B} \left( \pi - \alpha + \frac{1}{2} \sin 2\alpha \right)

= \frac{V_m}{2\pi} \left( \sin 2\alpha \right) + 2(\pi - \alpha) \frac{R_A}{R_A + R_B} \frac{R_B}{R_A + R_B}

Since \quad R_A = R_B = 10 \Omega, \quad V_m = 100 V \quad \text{and} \quad \alpha = 90^\circ,

a_1 = \frac{100}{2\pi} (\cos 180^\circ - 1) \times \frac{100}{10} = -1.59 A

b_1 = \frac{100}{\pi(10 + 10)} \left( \frac{\pi}{2} - \frac{1}{2} \sin 180^\circ \right) + \frac{100}{10} \left( \pi - \frac{\pi}{2} + \frac{1}{2} \sin 180^\circ \right)

100 \times \frac{\pi}{2} + \frac{100}{20\pi} \frac{\pi}{40} + \frac{100}{20} = 2.5 + 5 = 7.5 A

I_B = \frac{\sqrt{a_1^2 + b_1^2}}{\sqrt{2}} = \frac{\sqrt{(-1.59)^2 + (7.5)^2}}{\sqrt{2}} = 5.42 A

\psi = \tan^{-1} \frac{a_1}{b_1} = \tan^{-1} \frac{-1.59}{7.5} = -11.96^\circ

Although the load is resistive, a lagging \textit{PF} is introduced due to the chopping action. The reactive power here is fiction, since there is no energy storage element.

\textbf{Example 5.7}

A full-wave rectifier bridge supplying a 10 \Omega resistive load via an a.c. voltage controller from an ideal single-phase supply \( v_s = V_m \sin \omega t \) in an arrangement as shown Fig. 5.11. The SCRs of the inverse-parallel pair are gated to provide symmetrical phase-angle triggering.

(a) Sketch waveforms, for two cycles, of the supply voltage, load current and supply current.

(b) Derive an expression for the load power dissipation in terms of \( V_m, R \) and \( \alpha \). Calculate the power at \( \alpha = 30^\circ \) if \( V_m = 325 \text{ V} \).
(c) If diode $D_1$ is failed to open circuit, what will be the value of the load current?

![Fig.5.11.]

**Solution**

(a) The waveforms, for two cycles of the supply voltage, supply current and load current are shown in Fig.5.12.

![Fig.5.12. Waveforms.]

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(b) \( V_L = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} (V_m \sin \omega t)^2 d\omega t} = \sqrt{\frac{1}{\pi} \int_\alpha^{\pi} V_m^2 \sin^2 \omega t d\omega t} \)

\[ = \frac{V_m^2}{2\pi} \int_\alpha^{\pi} (1 - \cos 2\omega t) d\omega t = \sqrt{\frac{V_m^2}{2\pi} \left[ 1 - \frac{\alpha}{\pi} + \frac{1}{2\pi} \sin 2\alpha \right]} \]

Power

\[ P = \frac{V_L^2}{R} = \frac{V_m^2}{2\pi R} \left[ 1 - \frac{\alpha}{\pi} + \frac{1}{2\pi} \sin 2\alpha \right] \]

For \( \alpha = 30^\circ \Rightarrow \frac{\pi}{3} \) and \( V_m = 325 \text{ V} \)

\[ P = \frac{(325)^2}{2 \times 10} \left[ 1 - \frac{\pi/6}{\pi} + \frac{1}{2\pi} \sin 60^\circ \right] = 5371 \text{ W} \]

(c) If \( D_1 \) is failed to open circuit, the load current will be halved as shown in Fig.5.13, hence the power will be halved.

Fig.5.13 The load current waveform when \( D_1 \) failed to open circuit.

**Example 5.8**

A single-phase a.c. controller shown in Fig.5.1 is used to control power flow to a resistive load \( R \) from an ideal sinusoidal supply \( v_s = V_m \sin \omega t \). The SCRs are gated to produce symmetrical angle triggering of the output voltage and current waveforms.

(a) Derive an expression for the load power dissipation \( P \) in terms of \( V_m, R \) and \( \alpha \). Calculate the power dissipation if the rms supply voltage is 230 V, \( R = 10 \Omega \), and \( \alpha = 90^\circ \)?

(b) The fundamental component of the current at \( \alpha = 90^\circ \) can be described by the expression
\[ i_{L1}(\omega t) = 0.65 \frac{V_m}{R} \sin(\omega t - 39.71^\circ) \]

Calculate the displacement factor, distortion factor and hence power factor of the circuit at \( \alpha = 90^\circ \).

**Solution**

(a) For single-phase a.c. controller, the *rms* value of the output voltage is given by Eq.(5.6) as,

\[ V_L = V_s \sqrt{\frac{1}{\pi} \left( \pi - \alpha + \frac{\sin 2\alpha}{2} \right)} \]

and the *rms* value of the load current is

\[ I_L = \frac{V_L}{R} = \frac{V_s}{R} \sqrt{\frac{1}{\pi} \left( \pi - \alpha + \frac{\sin 2\alpha}{2} \right)} \]

The power consumed by the load is

\[ P = V_L I_L \]

\[ = \frac{V_s^2}{R} \left[ \frac{1}{\pi} \left( \pi - \alpha + \left( \frac{\sin 2\alpha}{2} \right) \right) \right] \]

For \( V_s = 240 \, V \), \( R=25 \, \Omega \), \( \alpha = \frac{\pi}{2} \)

\[ \therefore P = \frac{(230)^2}{10} \left[ \frac{1}{\pi} \left( \pi - \frac{\pi}{2} + \frac{\sin 180^\circ}{2} \right) \right] = 2645 \, W \]

(b) The displacement factor \((\cos \psi_1)\) is

\[ i_{L1}(\omega t) = 0.65 \frac{V_m}{R} \sin(\omega t - 32.5^\circ) \]

\[ I_{L_{\text{max}}} = 0.65 \times \frac{V_m}{R} = 0.65 \times \frac{230 \times \sqrt{2}}{10} = 21.13 \, A \]

The *rms* value of the fundamental component of the current is

\[ I_{L1} = \frac{I_{L_{\text{max}}}}{\sqrt{2}} = \frac{21.13}{\sqrt{2}} = 14.95 \, A \]
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Example 5.9

The single-phase a.c. controller shown in Fig. 5.1 has a resistive load of $R = 15 \, \Omega$ and the input voltage is $V_s = 230 \, \text{V (rms)}$, 50 Hz. The firing angles of thyristors $T_1$ and $T_2$ are equal: $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, Determine:

(a) The $\text{rms}$ value of the output voltage.
(b) The input power factor.
(c) The average and $\text{rms}$ and values of the thyristor currents.

Solution

(a) Using Eq.(5.6),

$$V_m = \sqrt{2} \, V_s$$

$$V_L = \frac{V_m}{\sqrt{2}} \left[ \frac{1}{\pi} \left( \pi - \alpha + \frac{1}{2} \sin 2\alpha \right) \right]^\frac{1}{2}$$

For $\alpha = \frac{\pi}{2}$

$$V_L = 230 \left( \frac{1}{\pi} \left( \pi - \frac{\pi}{2} + \frac{\sin \pi}{2} \right) \right)^\frac{1}{2} = \frac{230}{\sqrt{2}} = 162.6 \, \text{V}$$
(b) The *rms* value of the load current is

\[ I_L = \frac{V_L}{R} = \frac{162.6}{15} = 10.85 \text{ A} \]

Load Power is

\[ P = I_L^2 R = (10.85)^2 \times 15 = 1763.86 \text{ W} \]

Because the input current is the same as the load current, the input VA rating is:

\[ \text{VA} = V_s I_s = 230 \times 10.85 = 2495.5 \]

The input *PF* is

\[ \text{PF} = \frac{V_L \cdot I_L}{V_s \cdot I_s} = \frac{V_L}{V_s} = \frac{162.6}{230} = 0.707 \]

(c) The average thyristor current

\[ I_{T(avg)} = \frac{1}{2\pi R} \int_{\alpha}^{\pi} V_m \sin \omega t \, d\omega \]

\[ = \frac{V_s \sqrt{2}}{2\pi R} (\cos \alpha + 1) = \frac{\sqrt{2} \times 230}{2\pi \times 15} = 3.45 \text{ A} \]

The *rms* value of the thyristor current

\[ I_{T(rms)} = \sqrt{\frac{1}{2\pi} \int_{0}^{\pi} \left( \frac{\sqrt{2}}{R} V_s \sin \omega t \right)^2 \, d\omega} \]

\[ = \frac{V_s}{\sqrt{2}R} \left[ \frac{1}{\pi} (\pi - \alpha + \frac{\sin 2\alpha}{2}) \right]^{\frac{1}{2}} = \frac{230}{2 \times 115} = 7.66 \text{ A} \]

**Example 5.10**

A pair of thyristors connected in inverse-parallel, Fig.5.1, is used to supply adjustable current to a resistive load \( R = 12 \Omega \). If the supply voltage is \( v = \sqrt{2} \times 230 \sin \omega t \) and the thyristors are each triggered at an angle \( \alpha \) after their respective anode voltage zeros, it is required to:

(a) Calculate the power delivered to the load for \( \alpha = 30^\circ \).
(b) Determine the delay angle required to deliver 750 W to a 20 \( \Omega \) load.
(c) Calculate the \( \text{rms} \) of the source current and the \( \text{rms} \) and average value of the current in the thyristor for the angle obtained in (b) above.

Note that \( V_{L\text{normalized}} = \frac{V_L}{V_m/\sqrt{2}} \)

and the relation between the normalized \( \text{rms} \) load voltage and the delay angle \( \alpha \) for a single-phase a.c. voltage controller with a resistive load is given in Fig. 5.14.

![Fig. 5.14 Relation between output voltage and the triggering angle.](image)

**Solution**

(a) For a single-phase a.c. voltage controller, the \( \text{rms} \) value of the output voltage is given by Eq.(5.5) as

\[
V_L^2 = \frac{V_m^2}{2\pi} \left[ \pi - \alpha + \frac{\sin 2\alpha}{2} \right]
\]

For \( \alpha = 30^\circ = \frac{\pi}{6} \text{ rad.} \)

\[
V_L^2 = \frac{V_m^2}{2\pi} \left[ \frac{5\pi}{6} + 0.44330 \right] = 0.487 \ V_m^2
\]

\( \therefore \ V_L = 0.698 \ V_m \)

or \( V_L = 0.698 \times 230 \times \sqrt{2} = 227 \text{ V} \)

\( I_L = \frac{V_L}{R} = \frac{227}{12} = 18.9 \text{ A} \)
\[ P = \frac{V_L^2}{R} = \frac{(227)^2}{12} = 4294 \text{ W} \]

(b) The required voltage to deliver 750 W to a 20 Ω load is

\[ P = \frac{V_L^2}{R} \Rightarrow V_L = \sqrt{PR} = \sqrt{750 \times 20} = 122.5 \text{ V} \]

From the graph, the delay angle required to obtain a normalized output of \( \frac{122.5}{230} = 0.532 \) is approximately: \( \alpha = 110^\circ. \)

(a) Source current \( i_s \):

\[ I_s = I_L = \frac{V_L}{R} = \frac{122.5}{20} = 6.125 \text{ A} \]

The thyristor currents are determined as follows:

The \( rms \) value is given by,

\[ I_{SCR,rms} = \frac{I_L}{2} = \frac{6.125}{2} = 3.062 \text{ A} \]

The average value is given by,

\[ I_{SCR,(av)} = \frac{1}{2\pi} \int_{\alpha}^{\pi} \frac{V_m}{R} \sin \omega t \, d\omega t = \frac{V_m}{2\pi R} (1 + \cos \alpha) \]

\[ = \frac{\sqrt{2} \times 230}{2\pi \times 20} (1 + \cos 110^\circ) = 1.71 \text{ A} \]

5.5 OPERATION OF SINGLE-PHASE AC CONTROLLER WITH R-L LOAD

The single-phase a.c. voltage controller with resistive-inductive load is shown in Fig. 5.15. Due to the inductance in the circuit, the current in thyristor \( T_1 \) would not fall to zero at \( \omega t = \pi \), when the source voltage \( v_s \) start to be negative. Thyristor \( T_1 \) continues to conduct until its current \( i_L \) falls to zero at \( \omega t = \beta \).

The instantaneous supply voltage, \( v_s \), is now the sum of three components, consisting of the SCR voltage drop, \( v_T \), the resistance drop, \( v_R = iR \), and the voltage \( v_L \) across the inductor.

By KVL,

\[ v_s = v_R + v_L + v_T = v_s \quad (5.39) \]
Neglecting the voltage drop across the thyristor, the equation for the current through R-L load can be found from the solution of the differential equation:

\[ L \frac{di}{dt} + iR = V_m \sin \omega t \]  
(5.40)

The solution of this differential equation is:

The start of conduction is delayed until \( \omega t = \alpha \) subsequent to triggering, let the instantaneous current \( i(\omega t) \) consists of hypothetical steady-state components \( i_{ss}(\omega t) \) and transient component \( i_{tr}(\omega t) \), these analytical components of the current waveform are demonstrated in Fig 5.16. At the instant of SCR triggering the load current is zero. If the applied voltage \( v(\omega t) \) is given by Eq.(5.1) then the steady-state component of current \( i_{ss}(\omega t) \) in Fig.5.16 is given by:
Fig. 5.16 Analytical components of current waveform for single-phase a.c. voltage controller with series R-L load, $\theta = 60^\circ$, $\alpha = 120^\circ$.

$$i_{ss}(\omega t) = \frac{V_m}{Z} \sin(\omega t - \theta)$$

and the transient solution is

$$i_{tr}(\omega t) = Ae^{-\frac{t}{\tau}}$$

where

$$Z = \sqrt{R^2 + (\omega L)^2}$$

$$\tan \theta = \frac{\omega L}{R}$$

$$\tau = \frac{L}{R} = \text{time constant}$$

$$A = \text{constant}$$

The complete solution is: $i(\omega t) = i_{ss}(\omega t) + i_{tr}(\omega t)$

In the interval $\alpha < \omega t < \beta$, the total current is therefore described by the equation

$$i(\omega t) = \frac{V_m}{Z} \sin(\omega t - \theta) + Ae^{-\frac{t}{\tau}} \quad (5.41)$$
The constant $A$ can be found from initial conditions: when $\omega t = \alpha$, $i(\omega t) = 0$

\begin{align*}
0 &= \frac{V_m}{Z} \sin(\alpha - \theta) + Ae^{-\frac{\alpha}{\omega t}} \\
A &= -\frac{V_m}{Z} \sin(\alpha - \theta) e^{\frac{\alpha}{\omega t}} \quad (5.42)
\end{align*}

Substitute Eq.(5.42) into Eq.(5.41) yields

\begin{align*}
i(\omega t) &= \frac{V_m}{Z} \sin(\omega t - \theta) - \frac{V_m}{Z} \sin(\alpha - \theta)e^{\frac{\alpha}{\omega t}} e^{-\frac{\omega t}{\omega t}}
\end{align*}

The complete solution for the current at first cycle

\begin{align*}
i(\omega t) &= \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} [\sin(\omega t - \theta) - \sin(\alpha - \theta)e^{-\frac{(\omega t - \alpha)}{\omega t}}] \quad (5.43)
\end{align*}

In Fig. 5.16 for the part of the current cycle such that $\beta \leq \omega t \leq \pi + \theta$, the resultant current is mathematically negative. However, the conducting switch will not permit the flow of reverse current so that conduction ceases at point $\beta$, which is called the extinction angle or cut-off angle. The extinction angle $\beta$ can be obtained from Eq.(5.43) by noting that when $\omega t = \beta$, $i(\beta) = 0$. This results in the the following transcendental equation (5.44):

\begin{align*}
0 &= \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} [\sin(\beta - \theta) - \sin(\alpha - \theta)e^{-\frac{(\beta - \alpha)}{\omega L}}] \\
0 &= [\sin(\beta - \theta) - \sin(\alpha - \theta)e^{-\frac{(\beta - \alpha)}{\omega L}}] \\
\text{since} & \quad \frac{R}{\omega L} = \cot \theta \quad (5.44)
\end{align*}

If $\alpha$ and $\theta$ are known, $\beta$ can be calculated. However, this is a transcendental equation (i.e. cannot be solved explicitly and no way of obtaining the angle $\beta = f(\alpha, \theta)$). Method of solution is by iteration as discussed in Chapter Two. However, an approximate value of the extinction angle $\beta$ can be found from the simple relationship:

\begin{align*}
\beta &= 180^\circ + \theta - \Delta
\end{align*}
where

\[
\begin{align*}
\Delta &= 5^\circ \sim 10^\circ \text{ for large } R \text{ and small } \omega L \\
\Delta &= 10^\circ \sim 15^\circ \text{ for } R = \omega L \\
\Delta &= 15^\circ \sim 20^\circ \text{ for small } R \text{ and large } \omega L
\end{align*}
\]

For example, if \( \theta = 60^\circ \), \( \alpha = 120^\circ = 2\pi / 3 \), \( \cot \theta = 0.578 \)

Using Eq.(5.41) and solve by iterative method:

\[
\sin(\beta - 60^\circ) = \sin(120^\circ - 60^\circ)e^{-0.578(\beta - (2\pi/3))} \rightarrow \beta = 222^\circ
\]

Using the approximate solution:

\[
\begin{align*}
\theta &= 60^\circ, \quad \Delta = 15^\circ - 20^\circ \\
\beta &= 180^\circ + 60^\circ - 15^\circ = 225^\circ
\end{align*}
\]

or \( \beta = 180^\circ + 60^\circ - 20^\circ = 220^\circ \rightarrow \beta = \frac{225 + 220}{2} = 222.5^\circ \)

### 5.5.1 Load Voltage Waveform Analysis

The load voltage waveform for \( R-L \) load with a.c. voltage controller is shown in Fig.5.17.

![Load voltage waveform](image)

Fig.5.17 The load voltage waveform for \( R-L \) load.

This waveform can be represented as:

\[
\begin{align*}
\nu_L &= V_m\sin\omega t \left\{ \beta - \pi, \beta, 2\pi, \ldots \right\} \\
\nu_L &= 0 \quad \text{elsewhere}
\end{align*}
\]

\[ (5.45) \]
The rms value of the load voltage can be evaluated as

\[
V_L = \sqrt{\frac{2}{\pi} \int_{\alpha}^{\beta} (V_m \sin \omega t)^2 \, d\omega t}
\]

\[= \sqrt{\frac{V_m^2}{\pi} \int_{\alpha}^{\beta} \frac{1}{2} (1 - \cos 2\omega t) \, d\omega t}
\]

\[= V_m \left[ \frac{1}{2\pi} (\beta - \alpha + \frac{\sin 2\alpha}{2} - \frac{\sin 2\beta}{2}) \right]^{1/2}
\] (5.46)

Fourier analysis of the above waveform,

\[a_o = \frac{1}{\pi} \int_{0}^{2\pi} v_L(\omega t) \, d\omega t = 0
\] (5.47)

The fundamental component is given by

\[a_1 = \frac{1}{\pi} \int_{0}^{2\pi} v_L(\omega t) \cos \omega t \, d\omega t
\]

\[= \frac{1}{\pi} \int_{0}^{\beta-\pi} V_m \sin \omega t \cos \omega t \, d\omega t + \frac{1}{\pi} \int_{\alpha}^{\beta} V_m \sin \omega t \cos \omega t \, d\omega t
\]

\[+ \frac{1}{\pi} \int_{\alpha+\pi}^{2\pi} V_m \sin \omega t \cos \omega t \, d\omega t
\]

Use the trigonometric relation: \(\sin 2x = 2 \sin x \cos x\)

\[a_1 = \frac{V_m}{2\pi} \left[ \int_{0}^{\beta-\pi} \sin 2\omega t \, d\omega t + \int_{\alpha}^{\beta} \sin 2\omega t \, d\omega t + \int_{\alpha+\pi}^{2\pi} \sin 2\omega t \, d\omega t \right]
\]

\[a_1 = \frac{V_m}{4\pi} \left[ -\cos 2\omega t \left\{ \frac{\beta - \pi}{0} - \cos 2\omega t \left\{ \frac{\beta - \cos 2\omega t}{\alpha} \frac{2\pi}{\pi + \alpha} \right\} \right\}
\]

\[a_1 = \frac{V_m}{2\pi} \left[ \cos 2\alpha - \cos 2\beta \right]
\] (5.48)

Similarly

\[b_1 = \frac{1}{\pi} \int_{0}^{2\pi} v_L(\omega t) \sin \omega t \, d\omega t
\]
In the usual way, the peak value of the fundamental load voltage is given by

\[ c_1 = \sqrt{a_1^2 + b_1^2} \]

\[ c_1 = \frac{V_m}{2\pi} \sqrt{[\cos 2\alpha - \cos 2\beta]^2 + [2(\beta - \alpha) - \sin 2\beta + \sin 2\alpha]^2} \]  \hspace{1cm} (5.50)

The fundamental component of the load voltage will be phase displaced from the supply voltage by an angle \( \psi_{vl} \), which is different from the current displacement angle, hence

\[ \psi_{vl} = \tan^{-1} \frac{\cos 2\alpha - \cos 2\beta}{2(\beta - \alpha) - \sin 2\beta + \sin 2\alpha} \]  \hspace{1cm} (5.51)

Fourier coefficients for the \( n \)th higher harmonic of the load voltage waveform are found to be

\[ a_n = \frac{1}{\pi} \int_{0}^{2\pi} v_L(\omega t) \cos n\omega t \, d\omega t \]

\[ a_n = \frac{V_m}{2\pi} \left[ \frac{2}{n+1} \{\cos(n-1)\alpha - \cos(n-1)\beta\} + \frac{2}{n-1} \{\cos(n-1)\alpha - \cos(n-1)\beta\} \right] \]  \hspace{1cm} (5.52)

\[ b_n = \frac{1}{\pi} \int_{0}^{2\pi} V_L(\omega t) \sin n\omega t \, d\omega t \]
5.5.2 Harmonic Properties of the Current Waveform

The Fourier harmonic components of the current function $i(\omega t)$, equation (5.43), are found to be:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} i_L(\omega t) d\omega t = 0$$ (5.54)

The fundamental component of the current is found as

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} i(\omega t) \cos \omega t \, d\omega t$$

$$a_1 = \frac{V_m}{2\pi|Z|} \{ \cos(2\alpha - \theta) - \cos(2\beta - \theta) - \sin \theta(2\beta - 2\alpha)$$

$$+ 4\sin \theta \sin(2\alpha - \theta) - \{ \cos(\theta + \beta)e^{\cot \theta(\beta - \alpha)} - \cos(\theta + \alpha) \} \}$$ (5.55)

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} i(\omega t) \sin \omega t \, d\omega t$$

$$b_1 = \frac{V_m}{2\pi|Z|} \{ \sin(2\alpha - \theta) - \sin(2\beta - \theta) + \cos \theta(2\beta - 2\alpha)$$

$$+ 4\sin \theta \sin(\alpha - \theta) \{ \sin(\theta + \beta)e^{\cot \theta(\beta - \alpha)} - \sin(\theta + \alpha) \} \}$$ (5.56)

Variation of the fundamental component of the current with triggering angle is shown in Fig.5.18 for different values of phase-angle $\theta$. With purely resistive load, the exponential terms of the current equation (5.43) have zero value and the current wave consists of a sinusoid with symmetrical pieces chopped out. With a highly inductive load the current waveform is symmetrical about $\omega t=0$ and fundamental Fourier coefficient $b_1$ is therefore zero. By putting, $\theta = 90^\circ$ and $\beta = 2\pi - \alpha$ into Eq. (5.55) and Eq.(5.56), it is found that:

$$a_1 = \frac{-V_m}{\pi \omega L} [(\pi - \alpha) + \sin 2\alpha]$$ (5.57)

and

$$b_1 = 0$$ (5.58)
Fig. 5.18 Fundamental component of the current versus firing-angle for a single-phase a.c. voltage controller with series $R-L$ load.

For the $n^{th}$ harmonic Fourier coefficients $a_n$, $b_n$ are given, for general load phase-angle $\theta$, by

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} i(\omega t) \cos n\omega t \, d\omega t$$

$$b_n = \frac{1}{\pi} \int_{0}^{2\pi} i(\omega t) \sin n\omega t \, d\omega t$$

Solutions of the above two Fourier equations yield

$$a_n = \frac{V_m}{2\pi |Z|} \left\{ \frac{2}{n+1} \left[ \cos \{(n+1)\alpha - \theta\} - \cos \{(n+1)\beta - \theta\} \right] \\
- \frac{2}{n-1} \left[ \cos\{(n-1)\alpha - \theta\} - \cos\{(n-1)\beta - \theta\} \right] \\
+ \left[ (\cos n\beta \cot \theta - n \sin n\beta) e^{-\cot \theta (\beta - \alpha)} \right. \\
- \left. (\cos n\alpha \cot \theta - n \sin n\alpha) \frac{4 \sin(\alpha - \theta)}{n^2 + \cot^2 \theta} \right\}$$

(5.59)
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5.5.3 The rms Value of Load Current

The dissipation in a series circuit is proportional to the square of the total rms current. If the magnitudes of the steady-state rms current harmonics are now denoted by \( I_n \), the resultant rms current is given by

\[
b_n = \frac{V_m}{2\pi|Z|} \left\{ \sum_{n=1}^{\infty} \sin \left\{ (n+1)\alpha - \theta \right\} - \sin \left\{ (n+1)\beta - \theta \right\} \right. \\
- \frac{2}{n-1} \left[ \sin \{ (n-1)\alpha - \theta \} - \sin \{ (n-1)\beta - \theta \} \right] \\
+ \left[ (\sin n\beta \cot \theta - n \cos n\beta ) e^{-\cot \theta (\beta - \alpha)} \right] \\
- \left( \sin n\alpha \cot \theta - n \cos n\alpha \right) \frac{4 \sin (\alpha - \theta)}{n^2 + \cot \theta^2} \right\} \tag{5.60}
\]

For \( R-L \) loads the rms current can also be obtained from its defining integral

\[
I_{L_{rms}} = \sqrt{\int_{0}^{2\pi} i(\omega t)^2 \, d\omega t} \tag{5.62}
\]

Example 5.11

A resistive-inductive load is supplied from a 240 V, 50 Hz supply via inverse-parallel connected thyristors. If the load impedance is 20 \( \Omega \) and \( \theta = 40^\circ \) and the triggering angle is 60°, sketch the waveform for the current flowing through each thyristor and determine their conduction angle. Using the approximate method of finding \( \beta \).

Solution

Since \( \theta = 50^\circ > 45^\circ \), hence \( \Delta \approx 10^\circ - 15^\circ \)

The current waveform is shown in Fig.5.19.

Using approximate method:

\[
\beta = 180^\circ + \theta - \Delta \\
\beta_1 = 180^\circ + 50^\circ - 10^\circ = 220^\circ \\
\beta_2 = 180^\circ + 50^\circ - 15^\circ = 210^\circ
\]

Fig.5.19.
Conduction angle \[ \sigma = \beta - \alpha = 217.5^\circ - 60^\circ = 157.5^\circ \]

**Example 5.12**

A series \( R-L \) load of phase angle 45° is controlled by a pair of symmetrically triggered thyristors connected in inverse-parallel. Find an expression showing, approximately, the relationship between the fundamental load voltage and the thyristor firing angle.

**Solution**

The variation of the fundamental component with the firing angle for \( R-L \) load is given by:

\[ V_{1\text{rms}} = \frac{c_1}{\sqrt{2}} = \sqrt{\frac{a_1^2 + b_1^2}{2}} \]

From Eq. (5.47),

\[ c_1 = \frac{V_m}{2\pi} \sqrt{[\cos 2\alpha - \cos 2\beta]^2 + [2(\beta - \alpha) - \sin 2\beta + \sin 2\alpha]^2} \]

\[ \therefore V_{1\text{rms}} = \frac{V_m}{2\sqrt{2\pi}} \times \sqrt{[\cos 2\alpha - \cos 2\beta]^2 + [2(\beta - \alpha) - \sin 2\beta + \sin 2\alpha]^2} \]

To find \( \beta \), using the approximate method:

\[ \beta = 180 + \theta - \Delta \]

For \( \theta = 45^\circ \) \quad \Delta \approx 10^\circ

\[ \beta = 180^\circ + 45^\circ - 10^\circ = 215^\circ = 3.75 \text{ rads} \]

Let \( V_m = 1.0 \text{ p.u.} \) \quad \theta = 45^\circ = 0.785 \text{ rad}

\[ \therefore V_{1\text{rms}} = 0.1126 \times \sqrt{[\cos 2\alpha - 0.342]^2 + [2(3.75 - \alpha) - 0.939 + \sin 2\alpha]^2} \]

\[ = 4.05 \text{ p.u.} \]
Example 5.13

Two thyristors are connected in anti-parallel for the voltage control of a single-phase, series R-L load of phase angle $\theta$ to sinusoidal currents of supply frequency. If the thyristors are each triggered at angle after their respective anode voltage waveforms ($\alpha > \theta$).

(a) Calculate the extinction angle for $\theta = 60^\circ$ and $\alpha = 120^\circ$.
(b) Sketch the load voltage waveform and calculate its Fourier coefficients $a_1$, $b_1$, $c_1$, and $\psi_1$ for the fundamental (supply frequency) component.

Solution

(a) Finding $\beta$ : Using the approximate method,

$$\beta = 180 + \theta - \Delta \rightarrow \Delta = 15^\circ - 20^\circ$$

$$\beta_1 = 180^\circ + 60^\circ - 15^\circ = 225^\circ$$

$$\beta_2 = 180^\circ + 60^\circ - 20^\circ = 220^\circ$$

$$\therefore \beta = \frac{\beta_1 + \beta_2}{2} = \frac{225^\circ + 220^\circ}{2} = 222.5^\circ$$

(b) The load voltage waveform is as shown in Fig.5.17. The Fourier coefficients are

$$a_1 = \frac{V_m}{2\pi} (\cos 2\alpha - \cos 2\beta) = \frac{V_m}{2\pi} (-0.5 - 0.08) = -0.0923V_m$$

$$b_1 = \frac{V_m}{2\pi} [2(\beta - \alpha) - \sin 2\beta + \sin 2\alpha]$$

$$= \frac{V_m}{2\pi} \left[ 2 \left( \frac{222.5^\circ - 220^\circ}{180} \right) \pi - \sin 445^\circ + \sin 240^\circ \right] = 0.27V_m$$

$$c_1 = \sqrt{a_1^2 + b_1^2} = \sqrt{(-0.0923)^2 V_m^2 + (0.27)^2 V_m^2} \approx 0.28V_m$$

$$\psi_1 = \tan^{-1} \frac{a_1}{b_1} = \tan^{-1} \frac{-0.0923}{0.27} = -18.82^\circ$$

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Example 5.14

A pair of thyristors connected in inverse-parallel is controlled so as to gate at identical points of their respective anode voltages. This device controls the flow of current through a series $R-L$ load having a phase-angle of $30^\circ$. The supply voltage is given by $v = V_m \sin \omega t$.

(a) Sketch one cycle of the load voltage waveform $v(\omega t)$ for a firing angle $\alpha = 90^\circ$. Indicate on your diagram the extinction angle $\beta$ at the end of the completed conduction interval. Estimate, roughly, the conduction angle $\beta - \alpha$.

(b) Calculate, from first principles, an expression for the half-wave average value of the load voltage.

(c) The coefficient $a_1$ is given by

$$a_1 = \frac{V_m}{2\pi} (\cos 2\alpha - \cos 2\beta)$$

Derive an expression for the corresponding coefficient $b_1$ in terms of the firing angle $\alpha$ and current extinction angle $\beta$.

Solution

(a) $\theta = 30^\circ$, $\Delta = 5^\circ$

$$\beta = 180 + 30 - \Delta = 210^\circ - 5^\circ = 205^\circ$$

$$\sigma = \text{conduction angle} = \beta - \alpha = 205^\circ - 30^\circ = 175^\circ$$

(b) For half-wave, the average value is

$$V_{av(h.w)} = \frac{1}{2\pi} \int_0^{\beta - \pi} V_m \sin \omega t \, d\omega t + \frac{1}{2\pi} \int_{\alpha}^{\pi} V_m \sin \omega t \, d\omega t$$

$$= \frac{V_m}{2\pi} \left[ (-\cos \omega t) \right]_{0}^{\beta - \pi} + (-\cos \omega t) \left[ \frac{\pi}{\alpha} \right]$$

$$= \frac{V_m}{2\pi} [-\cos(\beta - \pi + 1) + 1 + \cos \alpha]$$

$$= \frac{V_m}{2\pi} [2 + \cos \alpha - \cos(\pi - \beta)]$$

$$= \frac{V_m}{2\pi} [2 + \cos 30^\circ - \cos 205^\circ] = 1.95 V_m$$
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5.6 THREE-PHASE AC-TO-AC VOLTAGE CONTROLLERS

5.6.1 Fully-Controlled Three-Phase AC Voltage Controller

In three-phase system, where large power is to be controlled, several possible circuit configurations for three-phase ac-to-ac voltage controllers are available. These various circuits are shown in Fig.5.20.

The circuits in Fig.5.20(a) and (b) are three-phase full-wave fully-controlled a.c. controller which are three-wire circuits and are difficult to analyse. In both these circuits, for current to flow, it is necessary to trigger at least two thyristors at a time one in each phase to get the controller started by establishing a current path between the supply lines. This necessitates two firing pulses spaced at 60° apart per cycle for firing each thyristor. The operation modes are defined by the number of thyristors conducting in these modes. The firing control range is 0° to 150°. The triplen harmonics are absent in both these configurations.

Let us consider first the three-wire circuit with a star-connected balanced resistive load as shown in Fig. 5.21. The analysis of operation of this three-phase a.c. controller can be understood by observing that the three-phase star-connected supply is applied to the star-connected resistive-load through two inverse-parallel thyristors in each phase. The six thyristors are turned on in the sequence $T_1-T_2-T_3-T_4-T_5-T_6$ at 60° intervals (one-sixth of the time period $T=2\pi/\omega$ of a complete cycle) and the gate signals are remain throughout the entire conduction angle.

For current to flow it is necessary to trigger at least two thyristors at a time. Hence at any interval, current flows in either three phases or two phases whereas the third phase is open, or no thyristor may be on and the instantaneous output voltages to the load are either line-to-neutral.
Fig. 5.20 Typical three-phase a.c. voltage controller circuit configurations.

voltages (three thyristors on), or one-half of the line-to-line voltage (two thyristors on) or zero (no thyristor on). The thyristors are triggered after a delay angle $\alpha$ from the natural commutation point. The natural commutation point is the starting of a cycle with $\omega t = 60^\circ$ of output voltage waveform, if six thyristors are replaced by diodes.
In the voltage controller of Fig. 5.21, depending on the firing angle \( \alpha \), there may be three operating modes as follows:

1. **Operating Mode-I: \( 0 \leq \alpha \leq 60^\circ \)**

   This mode is also known as mode 2/3. There are periods when three thyristors are conducting, one in each phase for, either direction or periods when just two thyristors conduct. To understand the operation of the three-phase controller in mode-I, consider the output phase voltage waveform \( v_{an} \) with \( \alpha = 30^\circ \) shown in Fig. 5.22. Assume that at \( \omega t = 0 \), the

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**Fig. 5.21** Three-phase a.c. voltage controller with star-connected \( R \)-load.

---

**Fig. 5.22** Operating Mode-I: \( 0 \leq \alpha \leq 60^\circ \), \( \alpha = 30^\circ \) with \( R \)-load.
two thyristors $T_5$ and $T_6$ are conducting, and the current through the $R$-load in phase-a is zero making $v_{an} = 0$. At $\omega t = 30^\circ$, $T_1$ is triggered by a gate pulse and starts conducting; $T_5$ and $T_6$ remain on and the output voltage is equal to the phase voltage $v_{an}$. The current in $T_6$ reaches zero at $60^\circ$, turning $T_6$ off. With $T_1$ and $T_6$ staying on, $v_{an} = 0.5 \ v_{AB}$. At $\alpha = 90^\circ$, $T_2$ is turned on, the three thyristors $T_1$, $T_2$, and $T_6$ are then conducting and the output voltage $= v_{AN}$. At $120^\circ$, $T_6$ turns off, leaving $T_1$ and $T_2$ on, so $v_{an} = 0.5 \ v_{AC}$. Thus with the progress of firing in sequence until or $\alpha = 60^\circ$, the number of the thyristors conducting at a particular instant alternates between two and three.

The instantaneous output phase voltage is therefore, depends on the number of conducting devices which, in turns, constructs its wave shape and it can be determined from Fig.5.22 as follows:

\[
\begin{align*}
v_{an} &= 0 & \text{For} & & 0 \leq \omega t < \pi/6 \\
v_{an} &= v_{AN} & \text{For} & & \pi/6 \leq \omega t < \pi/3 \\
v_{an} &= 0.5 \ v_{AB} & \text{For} & & \pi/3 \leq \omega t < \pi/2 \\
v_{an} &= v_{AN} & \text{For} & & \pi/2 \leq \omega t < 2\pi/3 \\
v_{an} &= 0.5 \ v_{AC} & \text{For} & & 2\pi/3 \leq \omega t < 5\pi/6 \\
v_{an} &= v_{AN} & \text{For} & & 5\pi/6 \leq \omega t < \pi \\
v_{an} &= 0 & \text{For} & & \pi \leq \omega t < 7\pi/6 \\
v_{an} &= v_{AN} & \text{For} & & 7\pi/6 \leq \omega t < 4\pi/3 \\
v_{an} &= 0.5 \ v_{AB} & \text{For} & & 4\pi/3 \leq \omega t < 3\pi/2 \\
v_{an} &= v_{AN} & \text{For} & & 3\pi/2 \leq \omega t < 5\pi/3 \\
v_{an} &= 0.5 \ v_{AC} & \text{For} & & 5\pi/3 \leq \omega t < 11\pi/6 \\
v_{an} &= v_{AN} & \text{For} & & 11\pi/6 \leq \omega t < 2\pi
\end{align*}
\]

The gating sequence can be generated by obtaining a pulse signal at the positive zero crossing of the supply phase voltage $v_{an}$. These pulses are, then, delayed by angles $\alpha$, $\alpha + \pi/3$ and $\alpha + 2\pi/3$ for gating $T_1$, $T_3$ and $T_5$. Similarly we have to generate pulses with delay angles $\pi + \alpha$, $4\pi/3 + \alpha$ and $5\pi/3 + \alpha$ for gating $T_2$, $T_4$, and $T_6$.

2. Operating Mode-II: $60^\circ \leq \alpha \leq 90^\circ$

This mode is also known as Mode 2/2 in which two thyristors, one in each phase, always conduct. For example when $\alpha = 75^\circ$ the output voltage waveform is as shown in Fig.5.23.

Just prior to or $\alpha = 75^\circ$, thyristor $T_5$ and $T_6$ were conducting and $v_{an} = 0$. At $\alpha = 75^\circ$, $T_1$ is turned on, $T_6$ continues to conduct while $T_5$ turns off as $v_{CN}$ is negative; $v_{an} = 0.5 \ v_{AB}$. When $T_2$ is turned on at $\alpha = 135^\circ$, $T_6$ is turned off and on and $v_{an} = 0.5 \ v_{AC}$. The next thyristor to turn on is $T_3$, which turns off $T_1$ and $v_{an} = 0$. One thyristor is always turned off when
another is turned on in this range of the triggering angle $\alpha$ and the output is either one-half line-to-line voltage or zero. The gating sequence can be generated by obtaining a pulse signal at the positive zero crossing of the supply phase voltage $v_{an}$. These pulses are, then, delayed by angles $\alpha$, $\alpha + 2\pi/3$, and $\alpha + 4\pi/3$ for gating $T_1$, $T_3$, and $T_5$. Similarly we have to generate pulses with delay angles $\pi + \alpha$, $5\pi/3 + \alpha$, and $7\pi/3 + \alpha$ for gating $T_2$, $T_4$, and $T_6$.

3. Operating Mode-III: $90^\circ \leq \alpha \leq 150^\circ$

This mode is also known as Mode 0/2 in which two thyristors conduct in two phases whereas the third phase is open. The output phase voltage, for example, is shown in Fig.5.24 for $\alpha = 120^\circ$.
For the interval $\alpha = 0$ to $\alpha = 120^\circ$, no thyristors were conduct and $v_{an} = 0$. At $\alpha = 120^\circ$, thyristor $T_1$, is triggered while $T_6$ has been already triggered. Since $v_{AB}$ is positive, $T_1$ and $T_6$ are forward-biased and they begin to conduct and $v_{an} = 0.5 v_{AB}$. Both $T_1$ and $T_6$ turn off when $v_{AB}$ becomes negative. When a gate signal is given to $T_2$, it turns on and $T_1$ turns on again. For $\alpha > 150^\circ$, there is no period when two thyristors are conducting and the output voltage is zero at $\alpha = 150^\circ$. Thus, the range of the firing angle control is $0^\circ \leq \alpha \leq 150^\circ$.

### 5.6.2 Analytical Properties of the Output Voltage Waveform

The procedure for obtaining an expression of the $rms$ value of the output voltage per phase for balanced star-connected $R$-load, which depends on the range of the triggering angle $\alpha$, as discussed before. If we define the instantaneous input phase voltages, taking $v_{an}$ as reference, as

$$
\begin{align*}
v_{AN} &= V_m \sin \omega t \\
v_{BN} &= V_m \sin (\omega t - 2\pi / 3) \\
v_{CN} &= V_m \sin (\omega t - 4\pi / 3)
\end{align*}
$$

(5.63)

then the instantaneous input line-to-line voltages are

$$
\begin{align*}
v_{AB} &= \sqrt{3} v_{AN} = \sqrt{3} V_m \sin (\omega t + \pi/6) \\
v_{BC} &= \sqrt{3} v_{BN} = \sqrt{3} V_m \sin (\omega t - \pi/2) \\
v_{CA} &= \sqrt{3} v_{CN} = \sqrt{3} V_m \sin (\omega t - 7\pi/6)
\end{align*}
$$

(5.64)

The $rms$ values of the output voltage waveform per phase for $R$-load for the three ranges of the triggering angle are found using Eq.(5.4) as follows:

For $0 \leq \alpha \leq 60^\circ$,

$$
V_L = \sqrt{3}V_m \left[ \frac{2}{2\pi} \left[ \int_{\alpha}^{\pi/3} \frac{(\sin \omega t)^2}{3} d\omega t + \int_{\pi/2}^{\alpha+\pi/2} \frac{(\sin \omega t)^2}{4} d\omega t \\
+ \int_{\alpha+\pi/3}^{2\pi/3} \frac{(\sin \omega t)^2}{3} d\omega t + \int_{\pi/2}^{\alpha+\pi/2} \frac{(\sin \omega t)^2}{4} d\omega t \\
+ \int_{\alpha+2\pi/3}^{\pi} \frac{(\sin \omega t)^2}{3} d\omega t \right] \right]^{1/2}
$$

(5.65)
This leads to

\[ V_L = \sqrt{3}V_m \sqrt{\frac{1}{\pi} \left( \frac{\pi}{6} - \frac{\alpha}{4} + \frac{\sin 2\alpha}{8} \right)} \]  \hfill (5.66)

For \(60 \leq \alpha \leq 90^\circ\),

\[ V_L = \sqrt{3}V_m \left[ \frac{2}{2\pi} \left[ \int_{\alpha + \frac{\pi}{2} - \frac{\pi}{3}}^{\alpha + \frac{5\pi}{6} - \frac{\pi}{3}} \frac{(\sin \omega t)^2}{4} d\omega t + \int_{\alpha + \frac{\pi}{2} - \frac{\pi}{3}}^{\alpha + \frac{5\pi}{6} - \frac{\pi}{3}} \frac{(\sin \omega t)^2}{4} d\omega t \right] \right]^{1/2} \]

\[ V_L = \sqrt{3}V_m \sqrt{\frac{1}{\pi} \left( \frac{\pi}{12} - \frac{3\sin 2\alpha}{16} + \frac{\sqrt{3}\cos 2\alpha}{16} \right)} \]  \hfill (5.67)

For \(90 \leq \alpha \leq 150^\circ\),

\[ V_L = \sqrt{3}V_m \left[ \frac{2}{2\pi} \left[ \int_{\alpha + \frac{\pi}{2} - \frac{\pi}{3}}^{\pi} \frac{(\sin \omega t)^2}{4} d\omega t + \int_{\alpha + \frac{\pi}{2} - \frac{\pi}{3}}^{\pi} \frac{(\sin \omega t)^2}{4} d\omega t \right] \right]^{1/2} \]

\[ V_L = \sqrt{3}V_m \sqrt{\frac{1}{\pi} \left( \frac{5\pi}{24} - \frac{\alpha}{4} - \frac{\sin 2\alpha}{16} + \frac{\sqrt{3}\cos 2\alpha}{16} \right)} \]  \hfill (5.68)

where \(V_L = \text{rms load voltage per-phase.}\)

\[ V_m = \text{Peak value of the supply voltage per-phase.}\]

### 5.7 OTHER TYPES OF THREE-PHASE AC CONTROLLERS

Other three-phase configurations are shown in Fig.5.20 (c) and (d) that can be considered as three single-phase a.c. controllers working independently of each other and they are easy to analyse. In Fig.5.20 (c), the thyristors are to be rated to carry line currents and withstand phase voltages, whereas in Fig. 5.20 (d) they should be capable of carrying phase currents and withstand the line voltages. Also, in Fig. 5.20 (d) the line currents are free from triplen harmonics while these are present in the closed delta. The power factor in Fig.5.20 (d) is slightly higher. The Firing angle control range for both these circuits is \(0^\circ\) to \(180^\circ\) for \(R\)-load.
Another configuration is shown in Fig. 5.20 (e) when the controllers are delta connected and the load is connected between the supply and the converter. Here, current can flow between two lines even if one thyristor is conducting, so each thyristor requires one fixing pulse per cycle. The voltage and current ratings of the thyristors are nearly the same as those of the circuit in Fig. 5.20 (d).

It is also possible to reduce the number of thyristors to three and replace the other three thyristors by diodes in three-phase controller as shown in Fig. 5.20 (f). This controller is called thyrode controller or half-controlled a.c. controller.

**Three-Phase Delta-Connected AC Voltage Controller with Balanced Resistive Load**

The circuit of a three-phase, delta-connected a.c. voltage controller (termed as three-phase ac-to-ac voltage regulator) with balanced resistive load is shown in Fig. 5.25.

![Fig. 5.25 Three-phase delta-connected a.c. voltage controller with R-load.](image)

It may be noted that the resistance connected in all three phases are equal. Two thyristors connected back-to-back are used per-phase, thus needing a total of six thyristors. As stated earlier, the numbering scheme may be noted. It may be observed that one phase of the balanced circuit is similar to that used for single-phase a.c. voltage controller described in Section 5.1. Since the phase current in a balanced three-phase system is only \((1/\sqrt{3})\) of the line current, the current rating of the thyristors would be lower than that if the thyristors are placed in the line.

Assuming the line voltage \(V_{ab}\) as the reference, the instantaneous input line voltages are,
It may be noted that $V_m$ is the peak value of the line voltage in this case. The waveforms of the input line voltages, phase and line currents, and the thyristor gating signals, for $\alpha = 120^\circ$ are shown in Fig.5.26.

![Waveforms](image)

Fig.5.26 Waveforms for delta-connected three-phase a.c. voltage controller with $\alpha =120^\circ$: (a) Input line voltages, (b) Thyristors gate pulses, and (c) Load line currents.

**Example 5.15**

The three-phase a.c. voltage controller shown in Fig.5.21 is used to supply a star-connected resistive load of $R = 15$ $\Omega$. The line-to-line voltage is 400 V $rms$ and $f = 50$ Hz. For the delay $\alpha = \pi/6$ determine:

(a) The output power.
(b) The input power factor $PF$.
(c) Find an expression for the instantaneous phase load voltage.
Solution

(a) For $0^\circ \leq \alpha < 60^\circ$

$$V_L = \sqrt{3}V_m \sqrt{\frac{1}{\pi} \left( \frac{\pi}{6} - \frac{\pi}{6 \times 4} + \frac{sin2\alpha}{8} \right)}$$

$$V_m = \frac{400 \times \sqrt{2}}{\sqrt{3}} = 326.5 \text{ V}$$

$$V_L = \sqrt{3} \times 326.5 \times \sqrt{\frac{1}{\pi} \left( \frac{\pi}{6} - \frac{\pi}{6 \times 4} + \frac{sin60^\circ}{8} \right)} = 226.2 \text{ V}$$

$$I_L = \frac{V_L}{R} = \frac{226.2}{15} = 15.08 \text{ A}$$

$$P_L = 3I_L^2R = 3 \times (15.08)^2 \times 15 = 10.233 \text{ kW}$$

(b) Since the load is star-connection the phase current = line current. Thus the input volt-ampere power is

$$S = 3V_s I_L = 3 \times 230 \times 15.08 = 10.405 \text{ kVA}$$

$$PF = \frac{P_L}{S} = \frac{10.233}{10.405} = 0.983 \text{ lagging}$$

$$V_L = \sqrt{3} \times 326.5 \times \sqrt{\frac{1}{\pi} \left( \frac{\pi}{6} - \frac{\pi}{6 \times 4} + \frac{sin60^\circ}{8} \right)} = 226.2 \text{ V}$$

$$I_L = \frac{V_L}{R} = \frac{226.2}{15} = 15.08 \text{ A}$$

$$P_L = 3I_L^2R = 3 \times (15.08)^2 \times 15 = 10.233 \text{ kW}$$

(c) If we take phase – a for the input voltage as reference, $v_{AN} = 230\sqrt{2} \sin \omega t = 326.5 \sin \omega t$, the input line voltages are,

$$v_{AB} = 400 \sqrt{2} \sin (\omega t + \pi/6) = 565.6 \sin (\omega t + \pi/6)$$

$$v_{BC} = 565.6 \sin (\omega t - \pi/2)$$

$$v_{CA} = 565.6 \sin (\omega t - 7\pi/6)$$
For $\alpha = \pi/6$, the instantaneous load phase voltage is therefore,

\[
\begin{align*}
v_{an} &= 0 & \text{For } & 0 \leq \omega t < \pi/6 \\
v_{an} &= v_{AN} = 326.5 \sin \omega t & \text{For } & \pi/6 \leq \omega t < \pi/3 \\
v_{an} &= 0.5 \ v_{AB} = 282.8 \sin(\omega t + \pi/6) & \text{For } & \pi/3 \leq \omega t < \pi/2 \\
v_{an} &= v_{AN} = 326.5 \sin \omega t & \text{For } & \pi/2 \leq \omega t < 2\pi/3 \\
v_{an} &= 0.5 \ v_{AC} = 282.8 \sin(\omega t + 7\pi/6) & \text{For } & 2\pi/3 \leq \omega t < 5\pi/6 \\
v_{an} &= v_{AN} = 326.5 \sin \omega t & \text{For } & 5\pi/6 \leq \omega t < \pi \\
v_{an} &= 0 & \text{For } & \pi \leq \omega t < 7\pi/6 \\
v_{an} &= v_{AN} = 326.5 \sin \omega t & \text{For } & 7\pi/6 \leq \omega t < 4\pi/3 \\
v_{an} &= 0.5 \ v_{AB} = 282.8 \sin(\omega t + \pi/6) & \text{For } & 4\pi/3 \leq \omega t < 3\pi/2 \\
v_{an} &= v_{AN} = 326.5 \sin \omega t & \text{For } & 3\pi/2 \leq \omega t < 5\pi/3 \\
v_{an} &= 0.5 \ v_{AC} = 282.8 \sin(\omega t + 7\pi/6) & \text{For } & 5\pi/3 \leq \omega t < 11\pi/6 \\
v_{an} &= v_{AN} = 326.5 \sin \omega t & \text{For } & 11\pi/6 \leq \omega t < 2\pi
\end{align*}
\]

**PROBLEMS**

5.1 A single-phase full-wave a.c. voltage controller shown in Fig.5.1 has a resistive load of $R = 10 \ \Omega$ and the input voltage is $V_s = 120 \ V \ (rms)$, 50Hz. The firing angles of thyristors $T_1$ and $T_2$ are equal: $\alpha_1 = \alpha_2 = \frac{\pi}{2}$. Determine:

(a) The $rms$ value of the output voltage $V_L$.
(b) The input power factor.
(c) The average and $rms$ values of the thyristor currents.

[Ans: (a) 141.44 V, (b) 0.707 lagging , (c) $I_{Tav} = 4.5 \ A$, $I_{Trms} = 10 \ A$]
5.2 A pair of thyristors connected in inverse-parallel, as shown in Fig.5.27, is used to supply adjustable current to a resistive load of $R = 8 \, \Omega$. If the supply has an input voltage of 120 V (rms) and the thyristors are each triggered at an angle $\alpha$ after their respective anode voltage zeros. For $\alpha = 60^\circ$:

(a) Calculate the rms value $V_L$ of the load voltage and the rms value $I_L$ of the load current.
(b) Estimate the power absorbed by the load and the input power factor.
(c) Calculate the average and rms values of the thyristor current.
(d) Calculate the rms value of the fundamental component of the load voltage $V_{L1}$ and its phase angle $\psi_1$.

![Fig.5.27](image)

[Ans : (a) 107.6 V, 13.45 A, (b) 1447.62 W, 0.897 Lagging, (c) 5.06 A, 10.9 A, (d) 100.6 V, -16.57°]

5.3 A single-phase full-wave a.c. voltage controller consists of two thyristors connected in inverse-parallel as shown in Fig.5.1. The supply voltage is $V_s = 230 \, \text{V (rms)}$ at 50 Hz. The controller is used to regulate power to 1.5kW electric oven. Find the triggering angle $\alpha$ such that the power delivered to the oven is 0.5 kW.

[Ans: $\alpha = 74.5^\circ$]

5.4 A single-phase a.c. voltage controller consists of two thyristors connected in inverse-parallel as shown in Fig.5.1. The supply voltage is $V_s = 230 \, \text{V (rms)}$ at 50 Hz. The controller is used as a static step-down voltage transformer to control power transfer to a 20 $\Omega$ resistor. The triggering mode of the two thyristors is symmetrical with $\alpha = 60^\circ$.

(a) Sketch compatible waveforms for the supply, load and thyristor voltages.
(b) Calculate the rms value of the load voltage.
(c) Determine the input power factor of the circuit.
(d) Calculate the average value of the thyristor current.
(e) Explain why we have a power factor not unity while the load is pure resistance?

[Ans: (b) 207 V, (c) 0.896, (d) 10.31 A]

5.5 A pair of thyristors connected in inverse-parallel, as shown in Fig.5.1, is used to supply adjustable current to a resistive load of \( R = 20 \, \Omega \). If the supply voltage is \( v_s = 240 \sin \omega t \) and the thyristors are each triggered at an angle \( \alpha \) after their respective anode voltage zeros. For \( \alpha = 60^\circ \):

(a) Calculate the \( \text{rms} \) value \( V_L \) of the load voltage and the \( \text{rms} \) value \( I_L \) of the load current, hence estimate the power absorbed by the load.

(b) Calculate the \( \text{rms} \) value of the fundamental component of the load voltage \( V_{L1} \) and its phase angle \( \psi_1 \).

(c) Calculate the displacement factor, the distortion factor and the power factor.

(d) If thyristor \( T_2 \) is fail to operate, what effect would this have on the value of the output voltage of the converter at \( \alpha = 60^\circ \).

[Ans: (a) 152.27 V, 7.6 A, 1157 W, (b) 201.13 V, 142.2 V, -16.54\(^\circ\), (c) 0.958, 0.9358, 0.897, (d) If \( T_2 \) fails to operate, the circuit will behave as a half-wave controlled rectifier:

\[
V_{dc} = \frac{V_m}{2\pi} (1 + \cos \alpha) = \frac{240}{2\pi} (1 + \cos 60^\circ) = 57.32 \, V
\]

5.6 A single-phase full-wave a.c. voltage controller consists of two thyristors connected in inverse-parallel as shown in Fig.5.28. The supply voltage is \( V_s = 220 \, V \) (\textit{rms}) at 50 Hz. The controller is used to control heater element of 2500 W in an electric oven. The triggering mode of the thyristors is symmetrical with \( \alpha = 30^\circ \).

(a) Sketch compatible waveforms for the supply, load and thyristor voltages.

(b) Calculate the \( \text{rms} \) value of the load voltage and current.

(c) Determine the input power factor of the circuit.

(d) Calculate the average value of the thyristor current.

[Ans: (b) 216.8 V, 11.53 A, (c) 0.983]
5.7 A pair of thyristors connected in inverse-parallel is used to supply adjustable current to a resistive load of \( R = 10 \, \Omega \). If the supply voltage is 
\( v_s = 220 \sqrt{2} \sin \omega t \) and the thyristors are each triggered at an angle \( \alpha \) after their respective anode voltage zeros,

(a) Sketch the circuit arrangement of the converter.
(b) Prove that the \textit{rms} value \( V_L \) of the load voltage is given by

\[
V_L = V_m \sqrt{\frac{1}{2\pi} \left[ (\pi - \alpha) + \left( \frac{\sin 2\alpha}{2} \right) \right]}
\]

(c) Calculate the value of \( V_L \) for \( \alpha = 30^\circ \) and sketch its waveform.
(d) Calculate the \textit{rms} value \( I_L \) of the load current and the power delivered to the load.

[Ans: (c) 216.8 V, (d) 21.68 A, 4.7 kW]

5.8 A pair of thyristors connected in inverse-parallel is used to supply adjustable current to a resistive load of \( R = 6 \, \Omega \) in series with an inductance of \( L = 33 \, \text{mH} \). If the supply voltage is \( V_s \) is 240 V, 50 Hz and the thyristors are each triggered at an angle \( \alpha = 60^\circ \) after their respective anode voltage zeros, find (a) the conduction angle of each thyristor, (b) the \textit{rms} output voltage.

[Ans: (a) 162.3°, (b) 225.29 V]

5.9 A single-phase a.c. controller shown in Fig.5.1 is used to supply adjustable current to a resistive load of \( R = 10 \, \Omega \). If the supply voltage is \( v_s = 230 \sqrt{2} \sin \omega t \) and the thyristors are each triggered at an angle \( \alpha \) after their respective anode voltage zeros,

(a) Prove that the \textit{rms} value \( I_o \) of the adjustable load current is given by
\[ I_o = \frac{V_m}{\sqrt{2}R} \sqrt{1 - \frac{\alpha}{\pi} \sin 2\alpha} \]

(b) Calculate the value of \( I_o \) for \( \alpha = 45^\circ \) and sketch its waveform.
(c) Calculate the power delivered to the load and the power factor of the circuit.

[Ans: (b) 22 A, (c) 4796.1 W, 0.952]

5.10 A pair of thyristors connected in inverse-parallel, Fig.5.1, is used to supply adjustable current to a resistive load of 20 \( \Omega \). If the supply voltage is \( v_s = V_m \sin \omega t \) and the thyristors are each triggered at an angle \( \alpha \) after their respective anode voltage zeros,

(a) Sketch the load current waveform for \( \alpha = 60^\circ \).
(b) Obtain an expression for the \textit{rms} value \( I_L \) of the load current in terms of \( V_m, R \) and \( \alpha \).
(c) Calculate the value of \( I_L \) for \( \alpha = 60^\circ \) if \( V_m = 350 \text{ V} \).
(d) Calculate the power dissipated at the load for \( \alpha = 60^\circ \) and the power factor of the circuit.
(e) Calculate the fundamental (first harmonic) component of the load voltage when \( \alpha = 90^\circ \).

[Ans: (c) 11.1 A, (d) 2464.2 W, (e) 146.6 V]

5.11 A pair of thyristors connected in inverse-parallel as shown in Fig.5.1 is used to control the power flow to heating load from an ideal sinusoidal single-phase supply \( v_s(\omega t) = 325 \sin \omega t \). These thyristors are gated to produce symmetrical angle triggering of the current waveform.

(a) If the heating load is pure resistance of \( R = 20 \Omega \), calculate the \textit{rms} value of the output voltage and the power dissipation in the load at \( \alpha = 90^\circ \).
(b) The fundamental component of the load current at \( \alpha = 90^\circ \) can be described by the expression

\[ i_1(\omega t) = 0.83 \frac{V_s}{R} \sin(\omega t - 30^\circ) \]

where \( V_s = \text{rms} \) value of the supply voltage.
Calculate the displacement factor, distortion factor and hence the power factor of the circuit.
(c) If the load in (a) is replaced by a series resistor-inductor, estimate (do not calculate) the extinction angle of the current $\beta$ for $\theta = 50^\circ$ and $\alpha = 110^\circ$.

[Ans: (a) 162.5 V, 1320 W, (b) 0.843, 0.830, 0.700, (c) 212.5°]

5.12 The three-phase bidirectional a.c controller shown in Fig.5.21 supplies a star-connected resistive load of $R = 20 \, \Omega$ and the line-to-line input voltage is 415 V $rms$, 50 Hz. For firing angle $\alpha = 45^\circ$, determine (a) The $rms$ value of the output voltage per phase, (b) The line current, (c) The output power $P_o$, and (d) The input power factor of the controller.

[Ans : (a) $V_L = 222.6$ V, (b) $I_L = 11.13$ A, (c) $P_o = 7438$ W, (d) $PF = 0.929$ lagging.]

5.13 The three-phase bidirectional controller shown in Fig.5.21 supplies a star-connected resistive load of $R = 10 \, \Omega$ and the line-to-line input voltage is 400 V ($rms$), 50 Hz. Determine the $rms$ value of the output voltage and the power delivered to the load for the following triggering angles:
(a) $\alpha = 30^\circ$, (b) $\alpha = 75^\circ$ and (c) $\alpha = 120^\circ$.

[Ans: (a) $V_L = 225$ V, $P_L = 15190$ W, (b) $V_L = 162.66$ V, $P_L = 7937$ W, and (c) $V_L = 47.83$ V, $P_L = 6863.12$ W.]

5.14 The voltage of a three-phase heating load is to be controlled by using three-phase a.c. voltage controller as shown in Fig.5.21. The three-phase supply is star-connected with 400 V, 50 Hz line-to-line voltage. The load is 40 kW, (a) Find the current and voltage ratings of the thyristors assuming safety factor of 1.25 for both the current and voltage, (b) If the two inverse-parallel connected thyristors in each phase are replaced by triacs, what will be their current and voltage ratings?

[Ans: (a) For each thyristor: 36 A, 500 V, (b) For each triac: 72.15 A, 500 V]