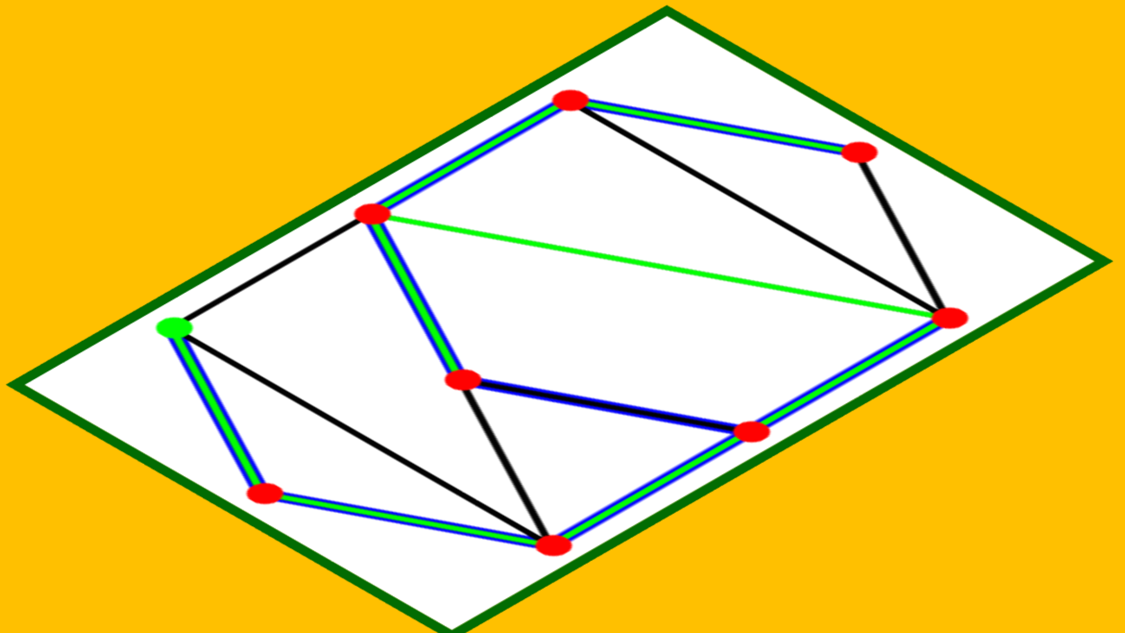
Chapter 6: Graphs



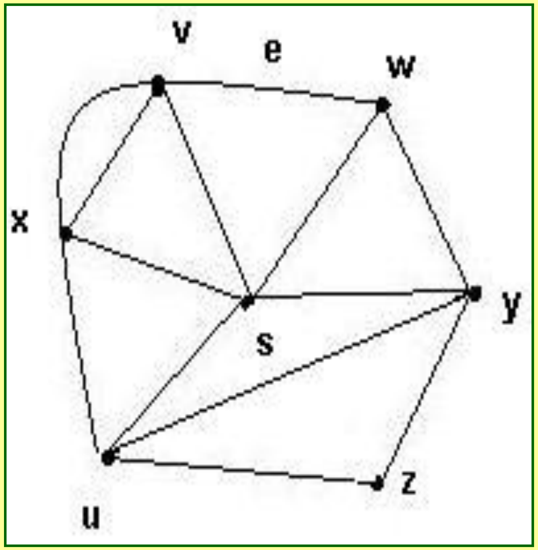
6.1 Graphs and Graph Models

Definition 1:

A **graph** $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

Summary

- It is a pair $G = (V, E)$, where
  - $V = V(G)$ = Set of vertices
  - $E = E(G)$ = Set of edges



- Example:
  - $V = \{s, u, v, w, x, y, z\}$
  - $E = \{(x, s), (x, v)_1, (x, v)_2, (x, u), (v, w), (s, v), (s, u),$
    $(s, w), (s, y), (w, y), (u, y), (u, z), (y, z)\}$

Example

This computer network can be modeled using a graph in which the vertices of the graph represent the data centers and the edges represent communication links.

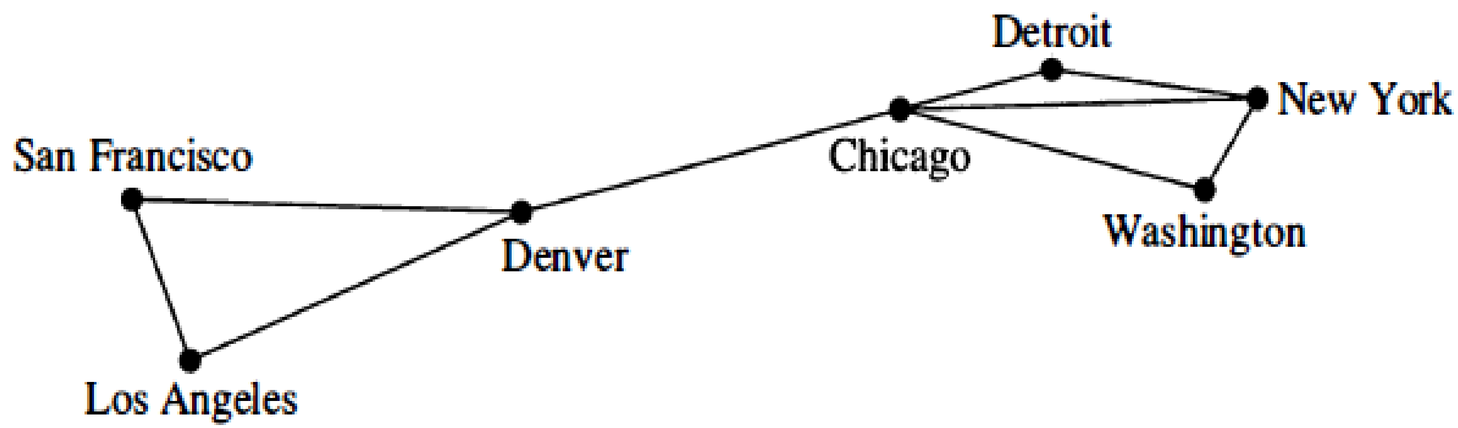


FIGURE 1 A Computer Network.

Graphs Types

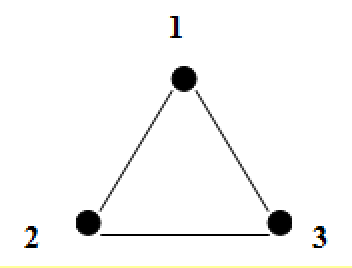
- simple graph.
- Multi - Graph
- Pseudo - graph
- Directed Graph
- Multi Directed Graph

Simple Graphs

A graph in which each edge connects two different vertices and where no more than one edge connect the same pair of vertices is called a **simple graph**.

E: (undirected edges)

e.g.



Multigraphs

A computer network may contain multiple links between data centers, as shown in Figure 2. To model such networks we need graphs that have more than one edge connecting the same pair of vertices. Graphs that may have multiple edges connecting the same vertices are called **multigraphs**.

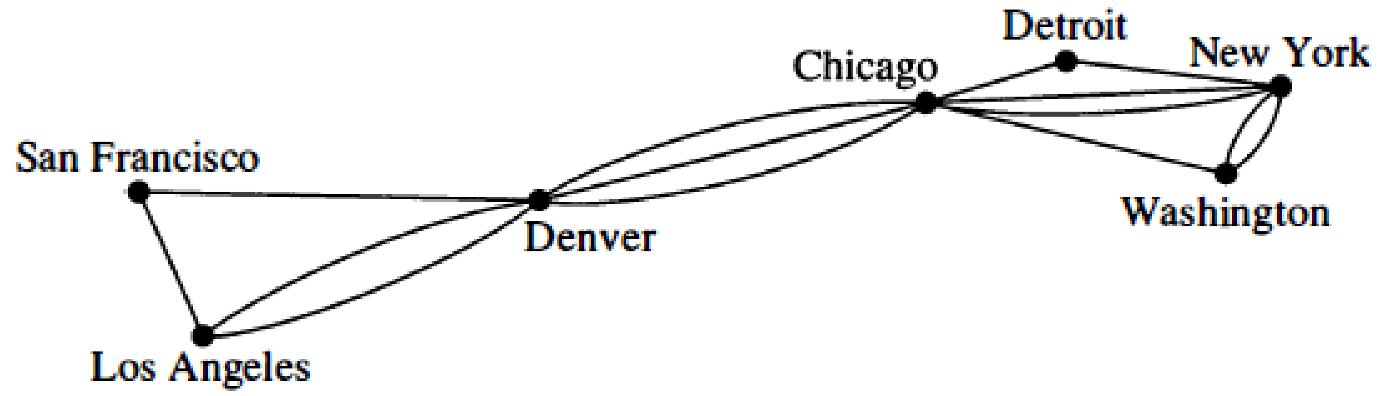


FIGURE 2 A Computer Network with Multiple Links between Data Centers.

Pseudographs

Sometimes a communication link connects a data center with itself. To model this network we need to include edges that connect a vertex to itself. Such edges are called **loops**. Graphs that may include loops, and possibly multiple edges are sometimes called **pseudographs**.

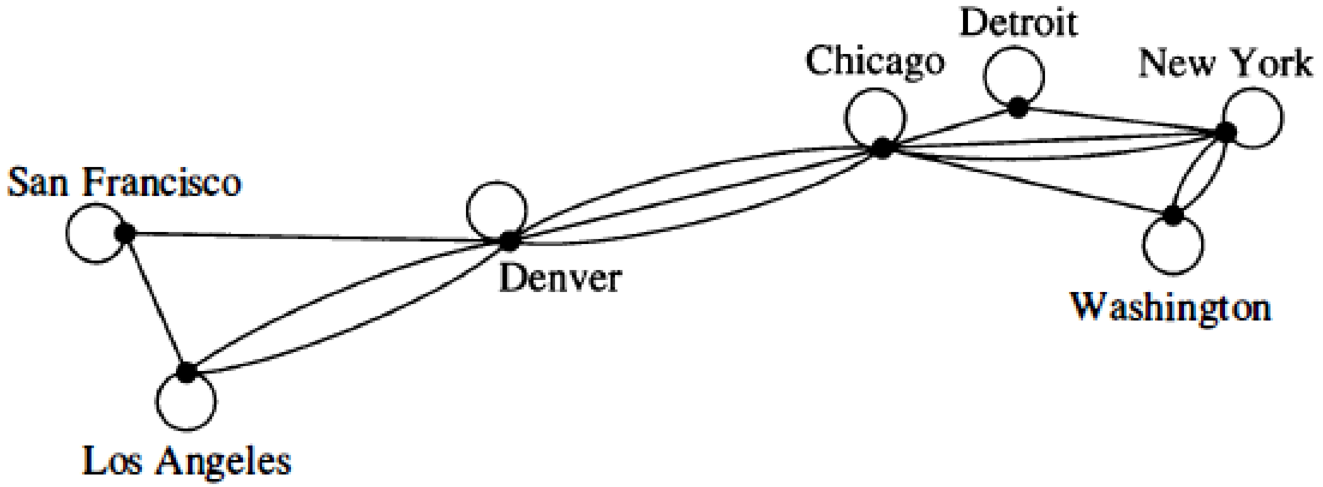


FIGURE 3 A Computer Network with Diagnostic Links.

Directed Graphs

Definition 2: A **directed graph** (or digraph) (V , E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices.

The directed edge associated with the ordered pair (u, v) is said to start at u and end at v. When a directed graph has no loops and has no multiple directed edges, it is called a **simple directed graph**.

Simple Directed Graph

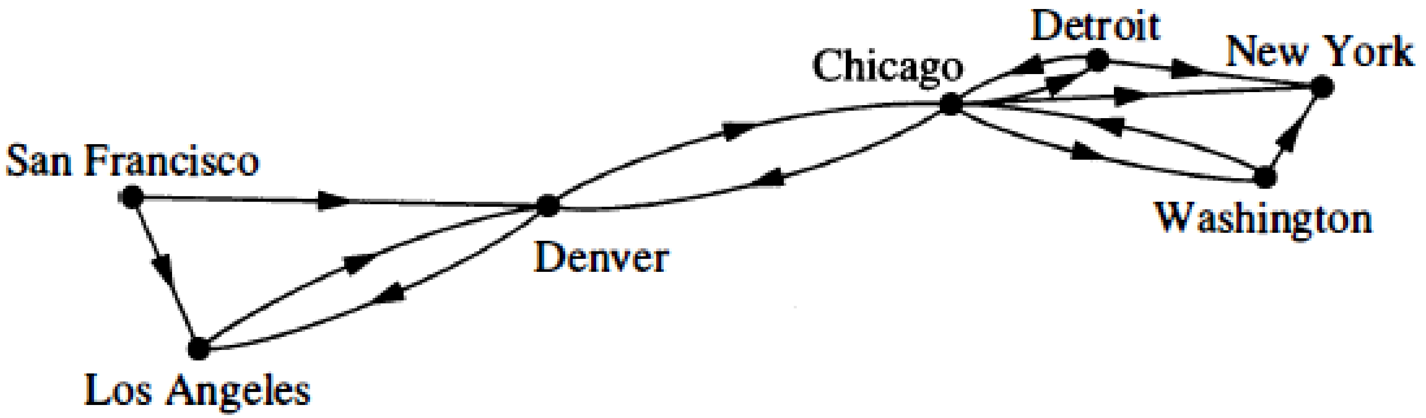


FIGURE 4 A Communications Network with One-Way Communications Links.

Directed Multigraph

Directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex are to used model such networks. We call such graphs **directed multigraphs**.

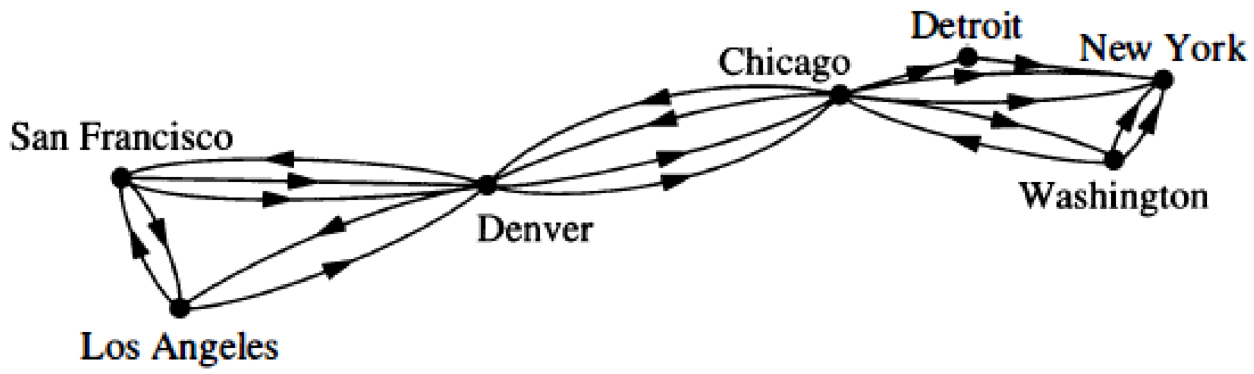


FIGURE 5 A Computer Network with Multiple One-Way Links.

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

6.2: Graph Terminology

- **Adjacent**, **connects**, **endpoints**, **degree**, **initial**, **terminal**, **in-degree**, **out-degree**, **complete**, **complete bipartite**, **cycles**, **wheels**, ***n*-cubes**.

Undirected Graph: Adjacency

Let G be an undirected graph with edge set E. Let $e \in E$ where $e = \{u, v\}$. We say that:

- The vertices u and v are **adjacent** / neighbors / connected.
- The edge e **connects** u and v.
- The vertices u and v are **endpoints** of the edge e.

Undirected Graph: Degree of a Vertex

- Let G be an undirected graph, $v \in V$ a vertex.
- The **degree** of v, $\deg(v)$, is its number of incident edges. (Except that any self-loops are counted twice.)
- A vertex with degree 0 is **isolated.**

Handshaking Theorem

- Let $G = (V, E)$ be an undirected (simple, multi- or pseudo-) graph with e edges. Then:

$$\sum_{v \in V} \deg(v) = 2|E|$$

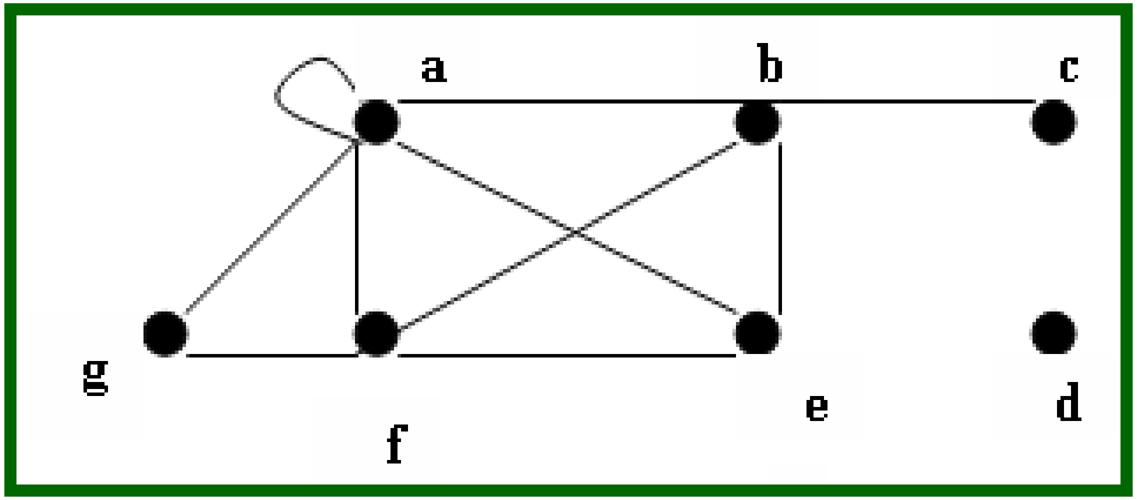
Example

How many edges are there in a graph with ten vertices each of degree 6?

Solution:

Sum of the degrees of the vertices $6 \times 10 = 60$.

Therefore: $60 = 2|E|$, $|E| = 30$



$\deg(a) = 6$

$\deg(b) = 4$

$\deg(c) = 1$ pendant

$\deg(d) = 0$ isolated

$\deg(e) = 3$

$\deg(f) = 4$

$\deg(g) = 2$

$$\sum \deg(v) = 20 = 2 \sum \text{edges} = 2\ (10)$$

Directed Graph: Adjacency

- Let G be a directed (possibly multi-) graph, and let e be an edge of G that is (u, v). Then we say:

$$u \longrightarrow v$$

  - u is **adjacent to** v, v is **adjacent from** u
  - e **comes from** u, e **goes to** v.
  - e **connects** u **to** v, e **goes from** u **to** v
  - The **initial vertex** of e is u
  - The **terminal vertex** of e is v

Directed Graph: Degree of a vertex

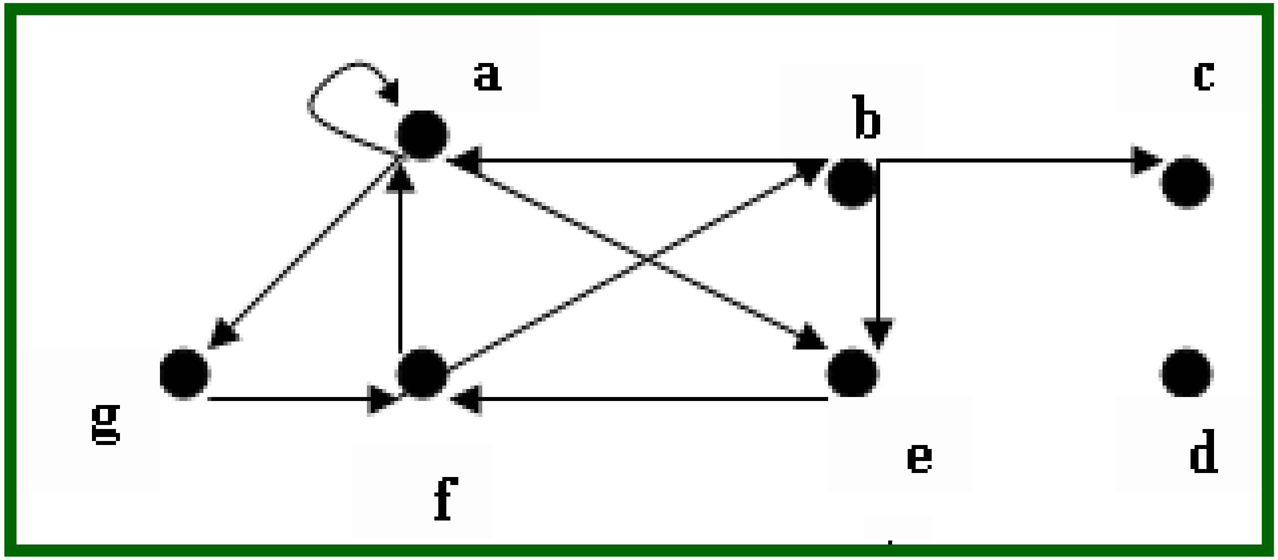
- Let G be a directed graph and v a vertex of G.
- The **in-degree** of v, $\deg^{-}(v)$, is the number of edges going to v (v is terminal).
- The **out-degree** of v, $\deg^{+}(v)$, is the number of edges coming from v (v is initial).
- The **degree** of v, $\deg(v) = \deg^{-}(v) + \deg^{+}(v)$, is the sum of v's in-degree and out-degree.
- A loop at a vertex contributes 1 to both in-degree and out-degree of this vertex.

Directed Handshaking Theorem

- Let G be a directed (possibly multi-) graph with vertex set V and edge set E. Then:

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = \frac{1}{2} \sum_{v \in V} \deg(v) = |E|$$

- Note that the degree of a node is unchanged by whether we consider its edges to be directed or undirected.



- $\deg^+(a) = 3$ $\deg^-(a) = 3$
- $\deg^+(b) = 3$ $\deg^-(b) = 1$
- $\deg^+(c) = 0$ $\deg^-(c) = 1$
- $\deg^+(d) = 0$ $\deg^-(d) = 0$
- $\mathrm{Deg}^+(e) = 1$ $\deg^-(e) = 2$
- $\mathrm{Deg}^+(f) = 2$ $\deg^-(f) = 2$
- $\mathrm{Deg}^+(g) = 1$ $\deg^-(g) = 1$

$$\sum \deg^+(v) = \sum \deg^-(v) = 1/2 \sum \deg(v) = \sum \text{edges} = 10$$

Special Graph Structures

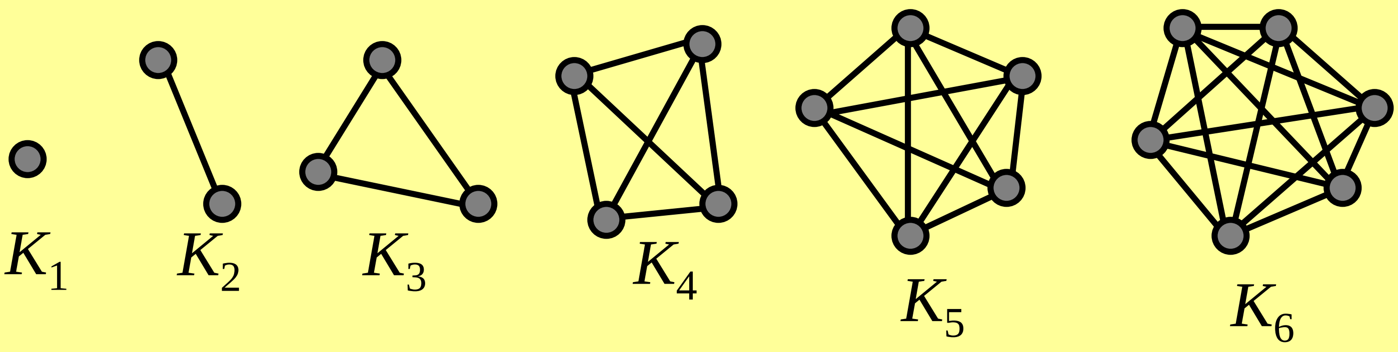
Special cases of undirected graph structures:

- Complete graphs: K_n
- Complete bipartite graphs: $K_{m,n}$
- Cycles: C_n
- Wheels: W_n

Complete Graphs

- For any $n \in \mathbf{N}$, a **complete graph** on n vertices, K_n, is a simple graph with n nodes in which every node is adjacent to every other node:

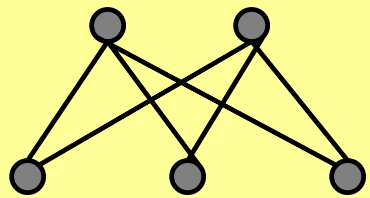
$$\forall u, v \in V: u \neq v \leftrightarrow \{u, v\} \in E.$$



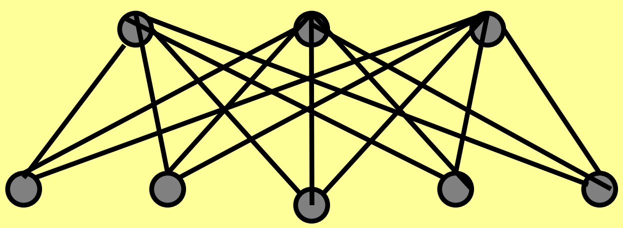
Note that K_n has n vertices and $\frac{n(n-1)}{2}$ edges.

Complete Bipartite Graphs

- The **complete bipartite** graph $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices such that there is an edge between vertices if and only if one vertix in the first set and the other vertix in the second set.



$K_{2,3}$



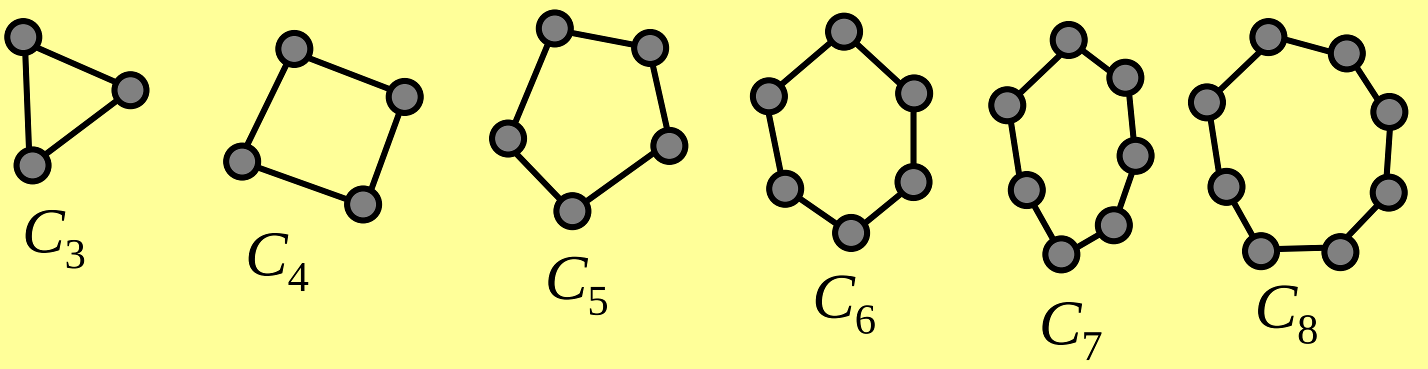
$K_{3,5}$

Note that $K_{m,n}$ has m+n vertices and mn edges.

Cycles

- For any $n \geq 3$, a **cycle** on n vertices, C_n , is a simple graph where $V = \{v_1,v_2, \ldots ,v_n\}$ and

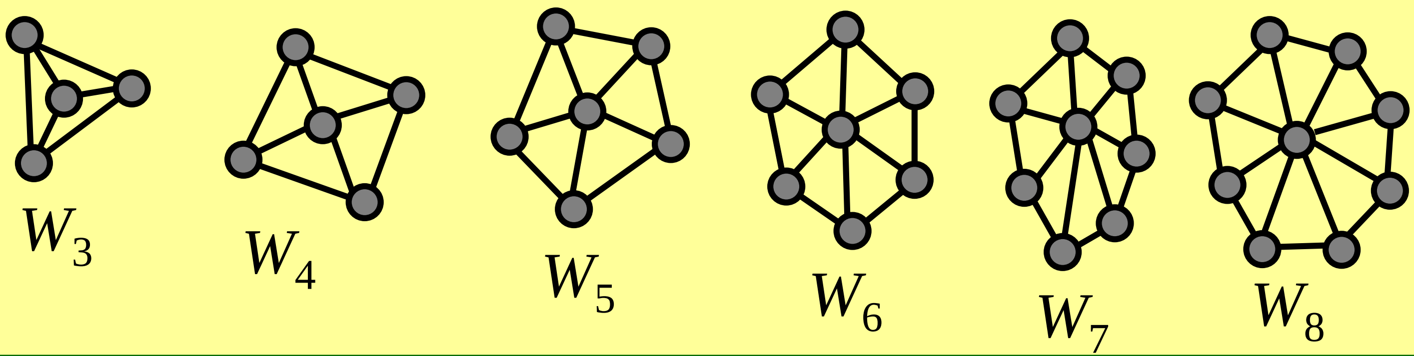
$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n , v_1\}\}.$$



Note that C_n has n vertices and n edges.

Wheels

- For any $n \geq 3$, a **wheel** W_n , is a simple graph obtained by taking the cycle C_n and adding one extra vertex v_{hub} and n extra edges $\{\{v_{\text{hub}}, v_1\}, \{v_{\text{hub}}, v_2\}, \ldots, \{v_{\text{hub}}, v_n\}\}$.



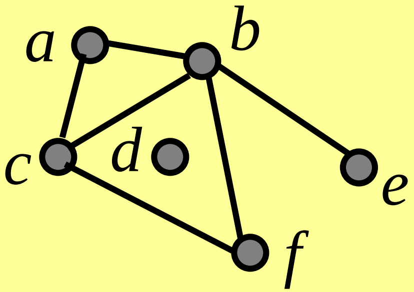
Note that W_n has n+1 vertices and 2n edges.

6.3 Graph Representations

- Graph representations:
  - Adjacency Lists.
  - Adjacency Matrices.

Undirected: Adjacency Lists

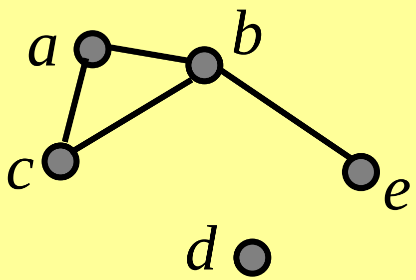
- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	Adjacent Vertices
a	*b*, *c*
b	*a*, *c*, *e*, *f*
c	*a*, *b*, *f*
d	
e	*b*
f	*c*, *b*

Undirected: Adjacency Matrices

- Matrix $A = [a_{ij}]$, where a_{ij} is 1 if $\{v_i, v_j\}$ is an edge of G, 0 otherwise.

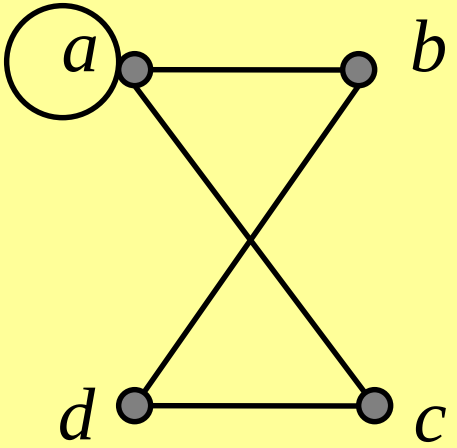


$$\begin{array}{c c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 1

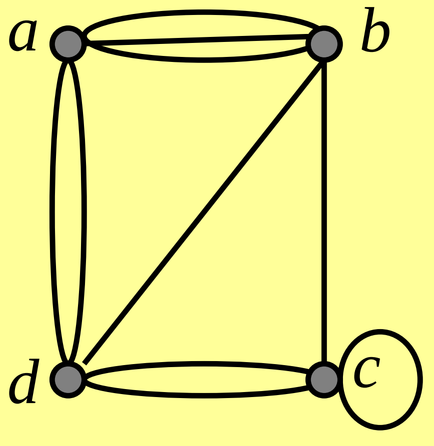
Draw a graph using the following adjacency matrix A with respect to the vertices: a, b, c, d :

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



Example 2

Find an adjacency matrix that represents the graph:



$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Example 3

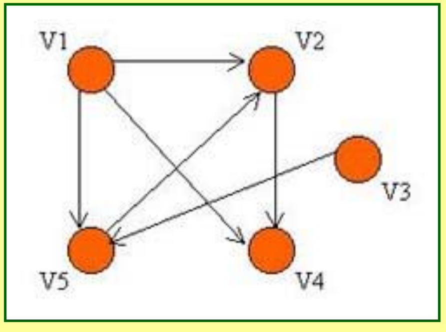
Let A be the adjacency matrix of a graph G with vertices a, b, c, d . Find the degrees of the vertices of G.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Answer: The degrees of the vertices a, b, c and d of G are 3, 2, 2 and 1, respectively.

Directed Adjacency Lists

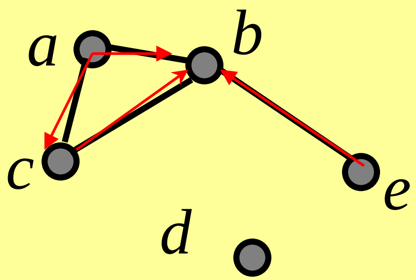
- 1 row per node, listing the terminal nodes of each edge incident from that node.



Initial vertex	Terminal vertices
V1	V2, V4, V5
V2	V4
V3	V5
V4	
V5	V2

Directed Adjacency Matrices

- Matrix $A = [a_{ij}]$, where a_{ij} is 1 if $\{v_i, v_j\}$ is an edge of G, 0 otherwise.

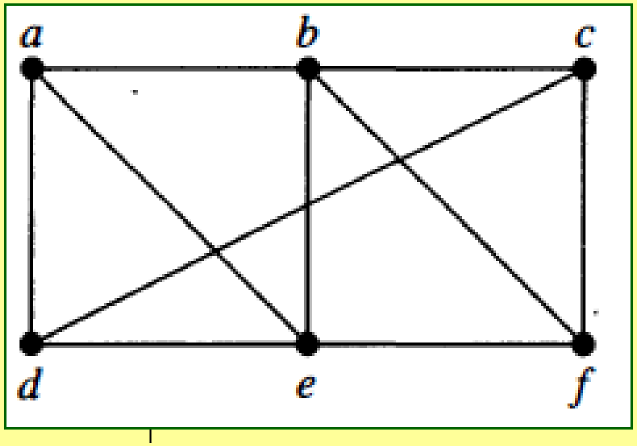


$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

6.4 Connectivity

- In an undirected graph, a **path of length *n*** from *u* to *v* is a sequence of adjacent edges going from vertex *u* to vertex *v*.
- A path is a **circuit** if $u = v$.
- A path **traverses** the vertices along it.
- A path is **simple** if it contains no edge more than once.

Example:



- **Simple path**: *a*, *d*, *c*, *f*, *e*

  The edges are {*a*, *d* }, {*d*, *c*}, {*c*, *f* }, {*f*, e}

  Length = 4
- **Circuit:** *b*, *c*, *f*, *e*, *b*

  The edges are {*b*, *c*},{*c*, *f* }, {*f*, *e*},{*e*, *b*}

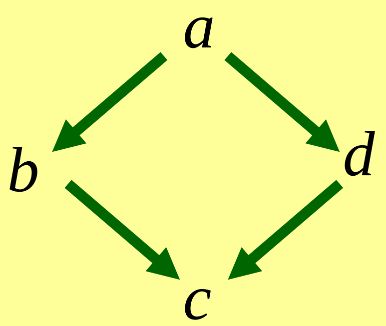
  Length = 4
- **Not a path :** *d*, *e*, *c*, *a* because {*e*, *c*} is not an edge
- **Not a simple path:** *a*, *b*, *e*, *d*, *a*, *b* because {*a*, *b*} appears twice

  Length = 5

Paths in Directed Graphs

- Same as in undirected graphs, but the path must go in the direction of the arrows.

e.g. Simple path *a*, *b*, *c*



Undirected Graphs: Connectedness

- An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

e.g.

$$\begin{tikzcd}[arrows=-] a \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & e \\ c \arrow[r] & d & \end{tikzcd}$$

Connected

$$\begin{tikzcd}[arrows=-] a \arrow[r] & b \arrow[d] & c \\ & d \arrow[r] & e \end{tikzcd}$$

Not connected

(No path between *a* and *c*)

Directed Graphs: Connectedness

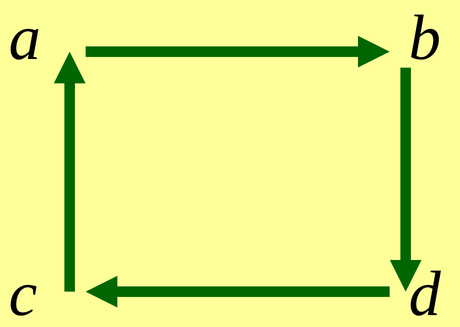
- Definition 1:

  A directed graph is **strongly connected** if there is a path from *a* to *b* and from *b* to *a* whenever *a* and *b* are vertices in the graph.

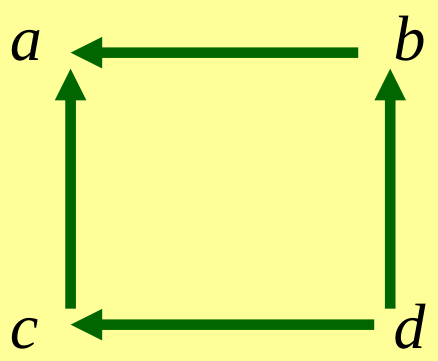
- Definition 2:

  A directed graph is **weakly connected** if there is a path between every two vertices

Example



Strongly connected



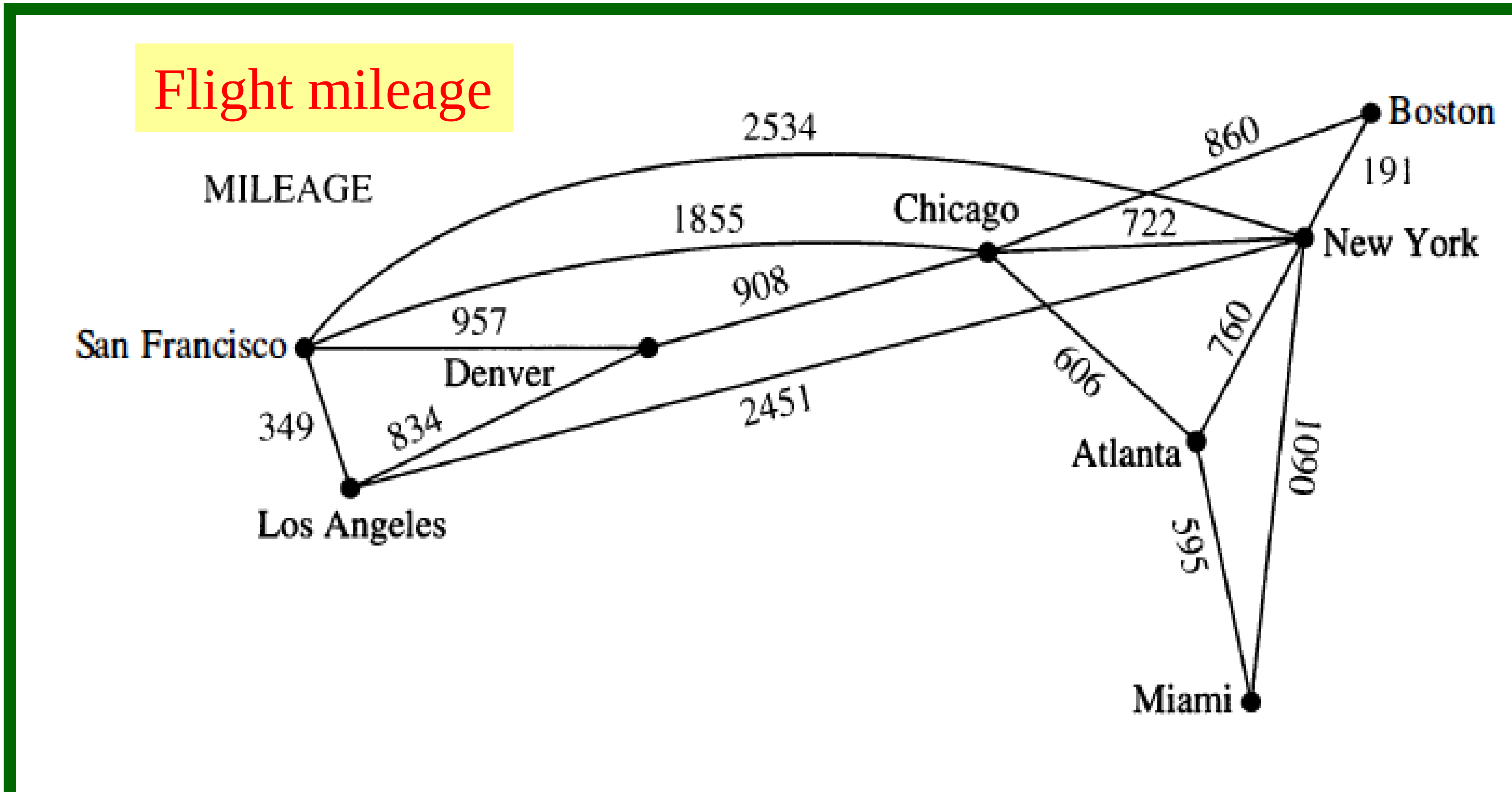
weakly connected

(No directed path from *a* to *b*)

9.6 Shortest-Path Problems

Many problems can be modeled using graphs with weights assigned to their edges. As an illustration, consider how an airline system can be modeled. We set up the basic graph model by representing cities by vertices and flights by edges. **Problems involving distances** can be modeled by assigning distances between cities to the edges. **Problems involving flight time** can be modeled by assigning flight times to edges. **Problems involving fares** can be modeled by assigning fares to the edges. Graphs that have a number assigned to each edge are called **weighted graphs**.

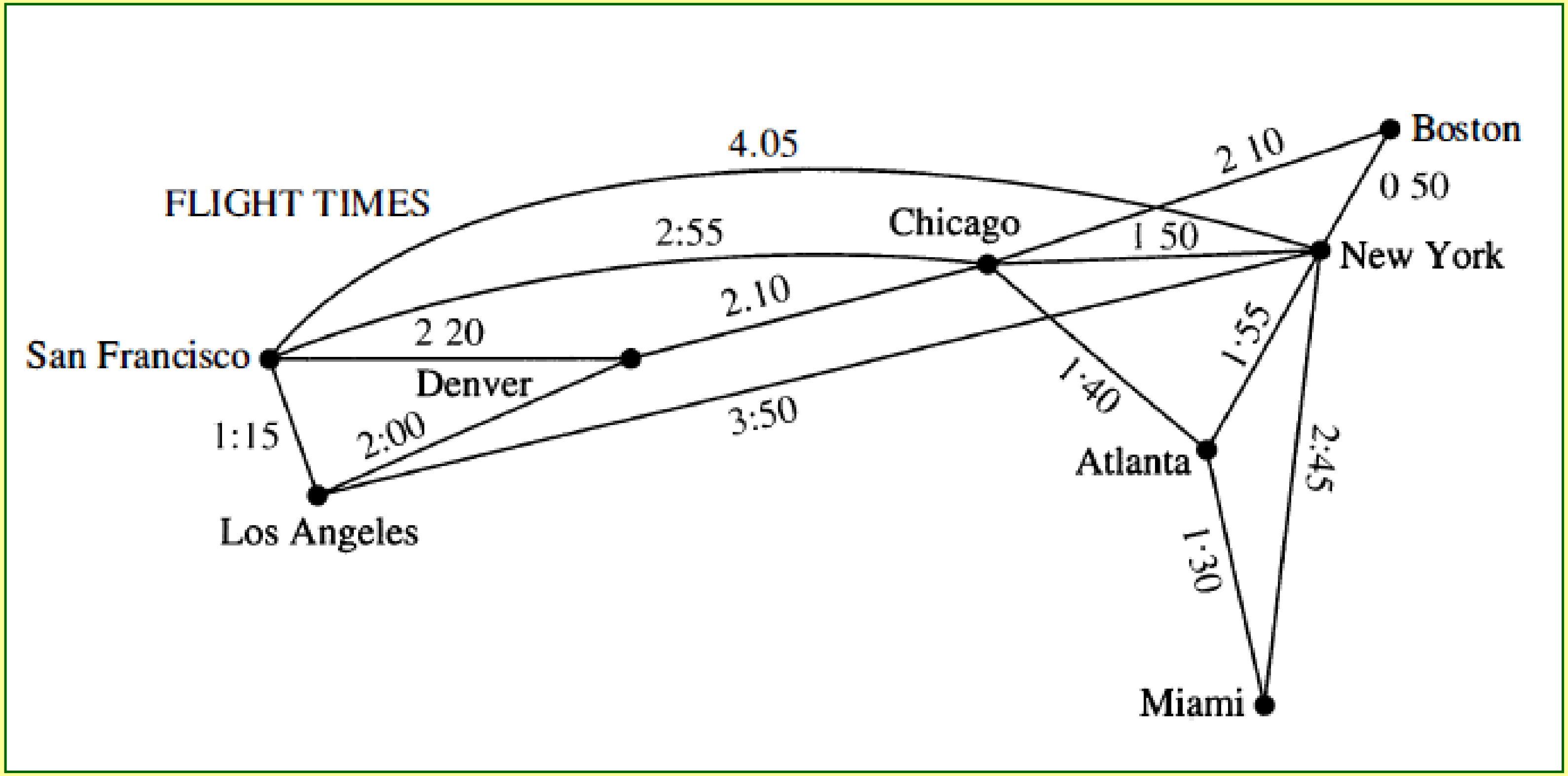
Example



Flight mileage
MILEAGE
2534
860
Boston
191
1855
Chicago
722
New York
908
957
San Francisco
Denver
760
606
2451
349
834
Atlanta
1090
Los Angeles
595
Miami

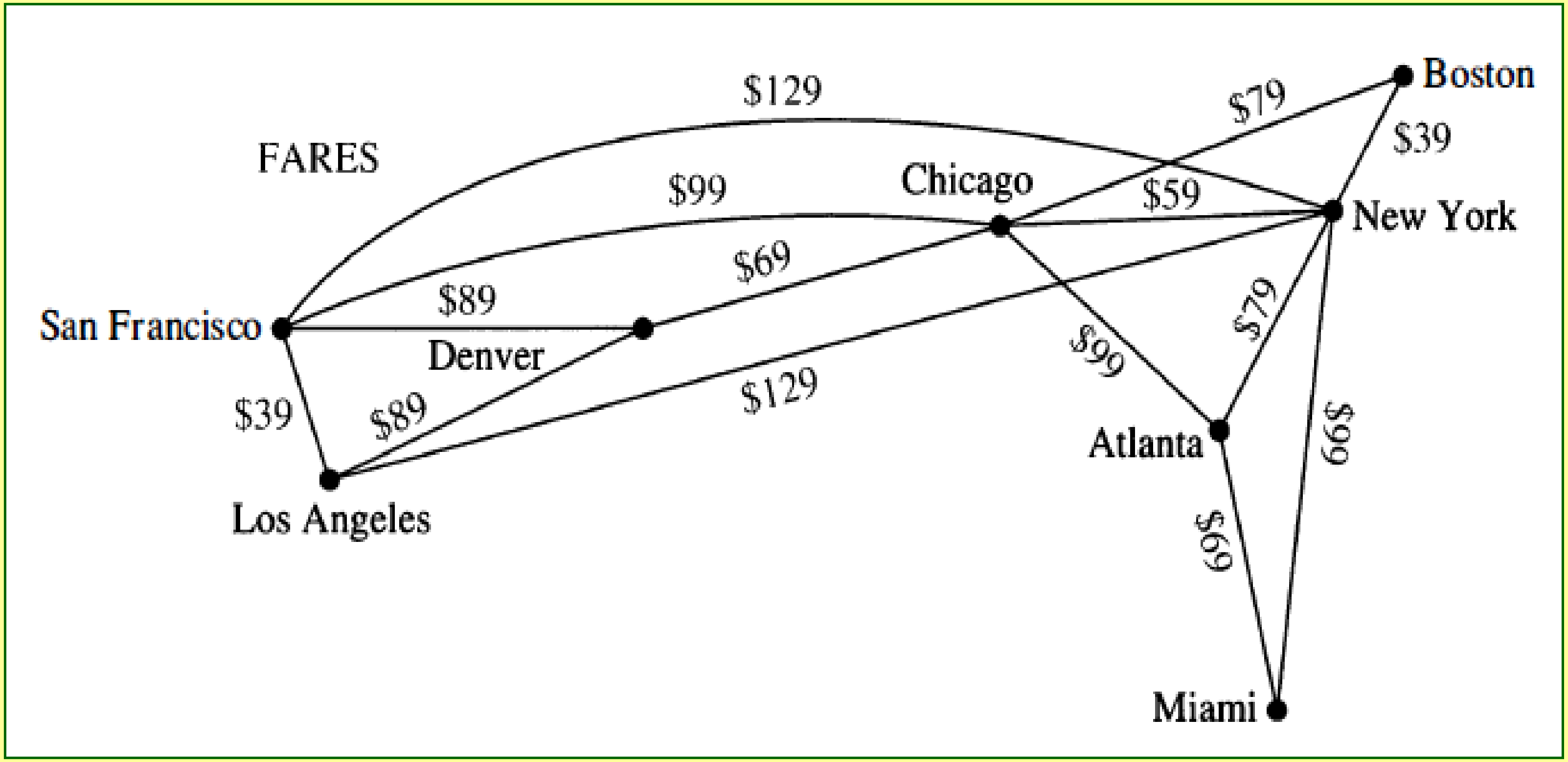
Example

Flight times



Example

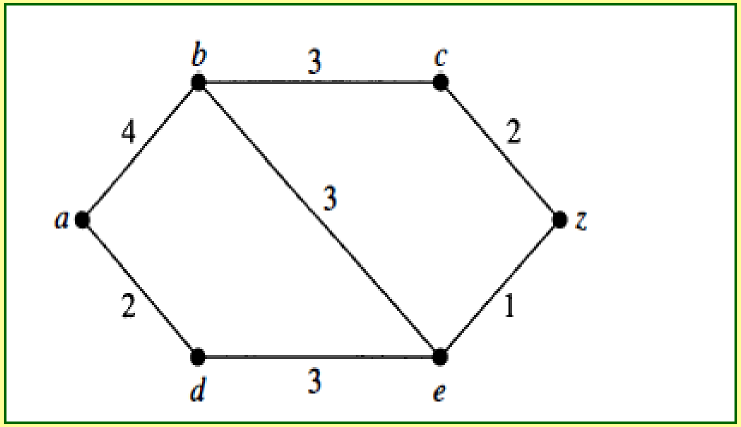
Flight Fares



Example

Shortest path, is a path of least length, between two given vertices

What is the shortest path from *a* to *z* in the figure ?



Path	length	
$a\ b\ c\ z$	$4 + 3 + 2 = 9$	
$a\ b\ e\ z$	$4 + 3 + 1 = 8$	
$a\ d\ e\ z$	$2 + 3 + 1 = 6$	(the shortest path)