Chapter :4

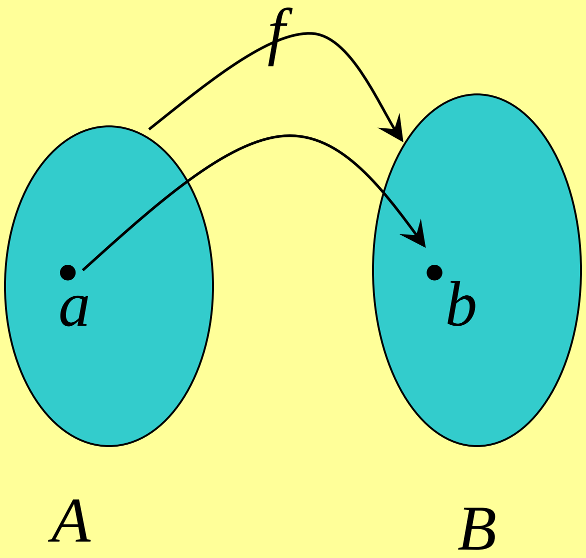
Functions

Function: Formal Definition

- Def. 1: Let A and B be sets, a function from A to B (f: A → B) is an assignment of exactly one elements of B to each element of A. where f(a) = b , and a $\in$ A, b$\in$B.

Graphical Representations

- Functions can be represented graphically in several ways:



Like Venn diagrams

Some Function Terminology

- If it is written that $f{:}A\rightarrow B$, and $f(a)=b$ (where $a\in A$ & $b\in B$), then we say:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - The *range* $R\subseteq B$ of f is $R=\{b \mid \exists a\, f(a)=b\}$.
  - If ***f*** is a function from A to B, we say that A maps B.

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Example

Let f: Z → Z such that

$$f(x) = x^2$$

The domain and the codomain is all integers.

The range of *f* is the set positive integers Z^+

Example

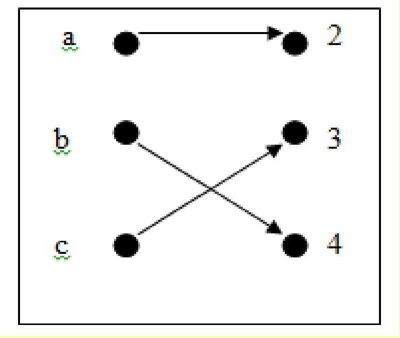
- let A={1,2,3} B={1,2,3,4,5}
- Determine whether the following is function from A $\rightarrow$B or not
- F1={(1,5), (2,2), (3,2)}   Yes
- F2={(1,2) ,(2,3) ,(2,5), (3,3)}   No

Function Representations

A function can be specified in different ways:

- Formula ex: $f(x) = x + 1$
- Graph: ex: function f: A → B

A = { a , b , c} and B = { 2 , 3, 4} $f(a) = 2$, $f(b) = 4$ $f(c) = 3$



- List: F={(a,2),(b,4),(c,3)}

Function Representations

- We can represent a function f:$A \rightarrow B$ as a set of ordered pairs $\{(a,f(a)) \mid a \in A\}$.
- Note that $\forall a$, there is only 1 pair (a,b).

Ex: function f: A → B

A = { a , b , c} and B = { 2 , 3, 4} f(a) = 2, f(b) = 4 f(c) = 3

F={(a,2),(b,4),(c,3)}

Function Operator Example

- +,× ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions* f,g:$\mathbf{R}\rightarrow\mathbf{R}$:
  - $(f+g)$:$\mathbf{R}\rightarrow\mathbf{R}$, where $(f+g)(x) = f(x) + g(x)$
  - $(f \times g)$:$\mathbf{R}\rightarrow\mathbf{R}$, where $(f \times g)(x) = f(x) \times g(x)$

Function Operator Example

- Ex: Let $f1$ and $f2$ be functions from R to R such that $f1(x) = x^2$ and $f2(x) = x - x^2$ what are the functions $f1 + f2$ and $f1\ f2$?

$(f1 + f2)(x) = f1(x) + f2(x) = x^2 + (x - x^2) = x$

$(f1\ f2)(x) = f1(x)\ f2(x) = x^2 * (x - x^2) = x^3 - x^4$

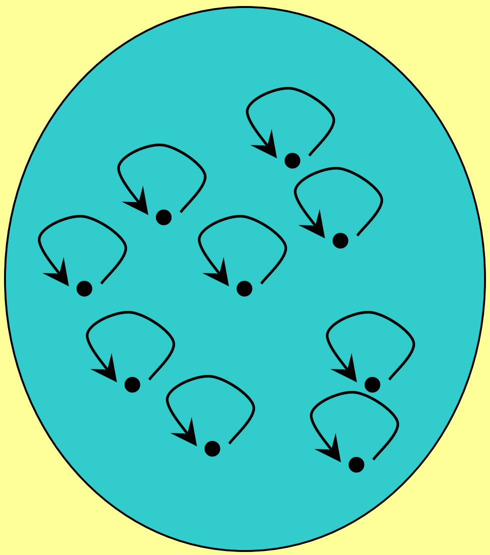
Identity function

Identity function on A is the function

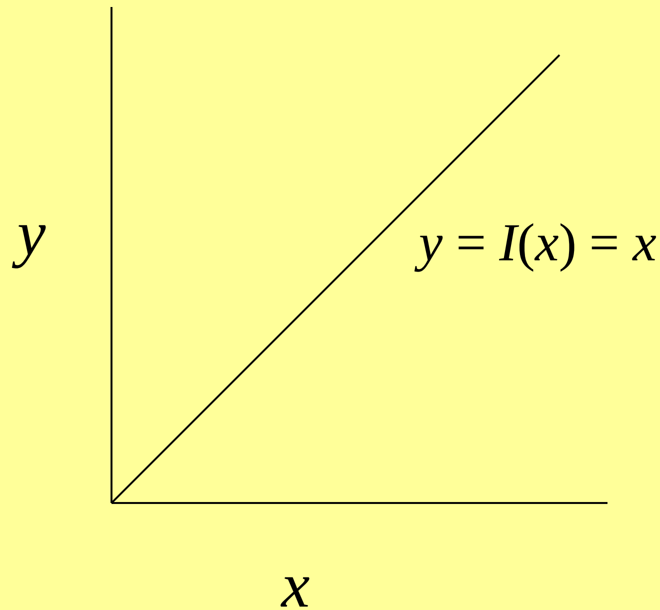
f:A → A , where f(x) = x

Identity Function Illustrations

- The identity function:



Domain and range



Functions Types

A. One –To - One (injective)

A function f is One- to –One if and only if $f(x) = f(y)$ implies that $x = y$ for all x and y in the domain of f.

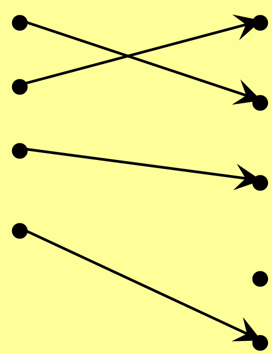
$\forall x \forall y \, (f(x) = f(y) \rightarrow x = y)$

or

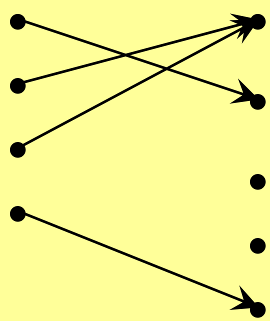
$\forall x \forall y \, (x \neq y \rightarrow f(x) \neq f(y))$

One-to-One Illustration

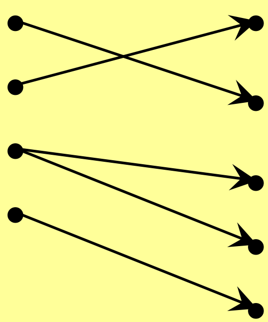
- (graph representations of functions that are (or not) one-to-one:



One-to-one



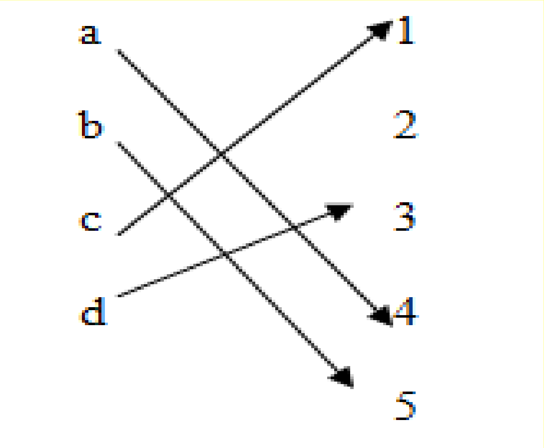
Not one-to-one



Not even a function!

Functions Types

Ex: The function *f* from {a, b, c, d} to {1, 2, 3, 4, 5} with *f*(a) = 4, *f*(b) =5 , *f*(c) = 1, and *f*(d) =3 is <u>one - to –one.</u>



Functions Types

Ex: Determine whether the functions $f: Z \rightarrow Z$ is one-to-one or not.

1. $f(x) = x^2$

Sol: The function $f(x) = x^2$ is not one-to-one because

$f(1) = f(-1) = 1$ but $1 \neq -1$

2. $f(x)=x+5$

$\Rightarrow$ it is one-to-one

Functions Types

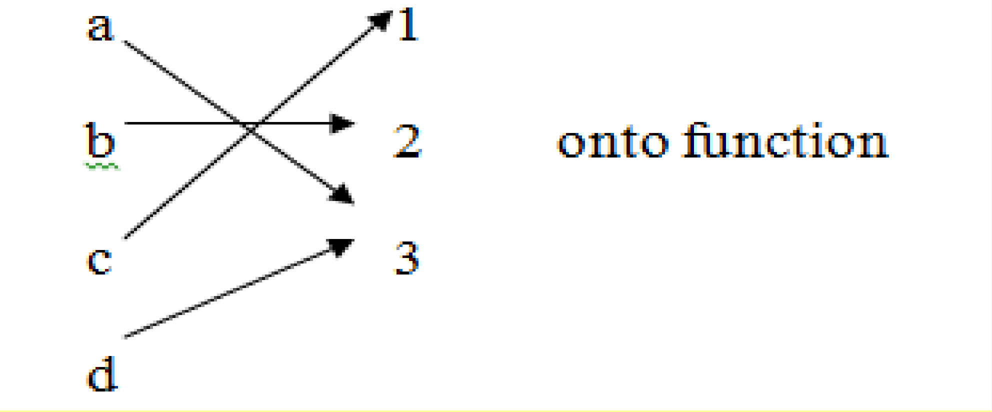
B. Onto (Surjective)

A function f:$A \rightarrow B$ is *onto* or *surjective* iff its range is equal to its codomain

$$(\forall b \in B, \exists a \in A: f(a)=b).$$

Functions Types

Ex: let the function *f* from {a, b, c, d} to { 1, 2 ,3} defined by *f*(a) = 3, *f*(b)= 2, *f*(c)=1, *f*(d) = 3, is *f* onto?



Functions Types

Ex: is the function $f(x) = x^2$ from Z → Z Onto function.

Sol.: It is not onto function, since there is no integer x such that $f(x) = -1$

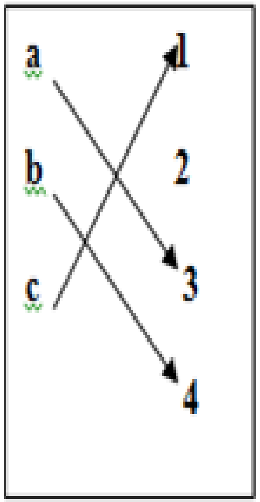
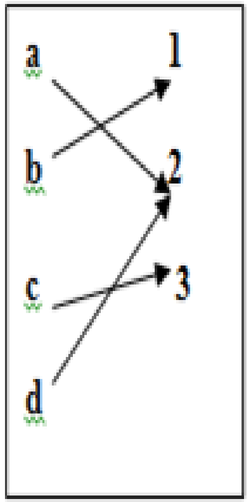
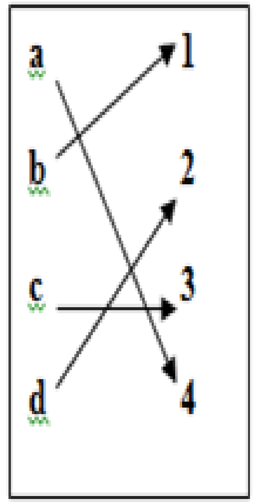
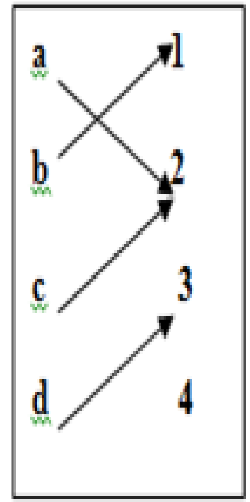
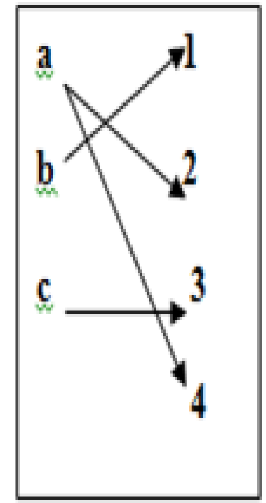
Functions Types

C. One- to –one correspondence (bijective)

A function is bijective if and only if it is both one-to-one and onto

Ex: identity function $f(x) = x$ is bijective

Functions Types

				
One-to-one Not Onto	Onto Not One-to-one	One-to-one Onto Bijective	Not Onto Not one-to-one	Not a function

Composition Function

Note match here.

- For functions $g:A\rightarrow B$ and $f:B\rightarrow C$, there is a special operator called *compose* ("$\circ$").
  - It composes (creates) a new function out of f and g by applying f to the result of applying g.
  - We say $(f\circ g):A\rightarrow C$, where $(f\circ g)(a) :\equiv f(g(a))$.
  - Note $g(a)\in B$, so $f(g(a))$ is defined and $\in C$.
  - is non-commuting. (Generally, $f\circ g \neq g\circ f$.)

Composition Function

Note match here.

Ex: Let $f(x) = 2x + 3$ and $g(x) = 3x + 2$ what is the composition of f and g if they both are from R to R?

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 6x + 7$

$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 6x + 11$

$\therefore (f \circ g)(x) \neq (g \circ f)(x)$

Inverse Function

For bijections $f{:}A\rightarrow B$, there exists an *inverse of f*, written $f^{-1}{:}B\rightarrow A$, where f(a) = b and f^{-1}(b) = a

which is the unique function such that

- (where I_A is the identity function on A)

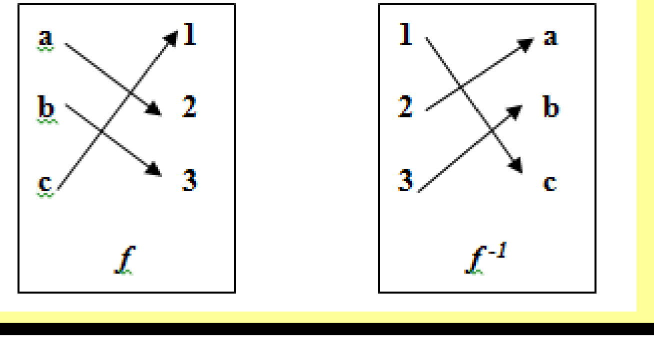
$$f^{-1} \circ f = I_A$$

Inverse Function

Ex: let f be the function from {a, b, c} to {1, 2, 3} such that $f(a)=2$, $f(b)=3$, $f(c)=1$, is f bijective and what is its inverse if it is?

Sol.: f is bijective because it 's one-to-one and onto

$f^{-1}(2)=a$, $f^{-1}(3)=b$, $f^{-1}(1)=c$



Inverse Function

- Ex: Let *f* be the function from Z to Z with $f(x) = x + 1$, is *f* bijective?

- Sol.: It is bijective and the inverse is

  $f(y) = y - 1$

  $\therefore \quad f^{-1}(x) = x - 1$

Inverse Function

EX: Find the inverse function of f(x)=3x+2

$$f^{-1}(x) = \frac{x-2}{3}.$$

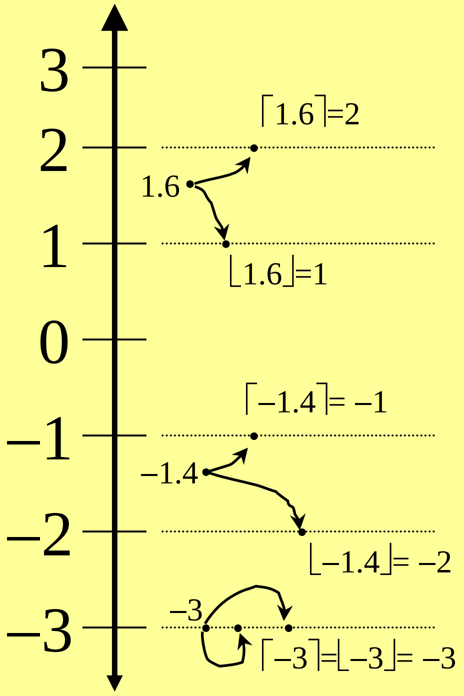
Floor And Ceiling Function

- *Floor* function $\lfloor x \rfloor$: the floor of real number x is the largest integer that is less than or equal to x.

- *Ceiling* function $\lceil x \rceil$: the ceiling of real number x is the smallest integer that is greater than or equal to x.

Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$, $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ & $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$, $\lfloor x \rfloor = \lceil x \rceil = x$.



Floor And Ceiling Function

- Ex: what is the value of the following?
- $\lfloor 1/2 \rfloor = 0$ $\lceil 1/2 \rceil = 1$
- $\lfloor -1/2 \rfloor = -1$ $\lceil -1/2 \rceil = 0$
- $\lfloor 3.1 \rfloor = 3$ $\lceil 3.1 \rceil = 4$
- $\lfloor 7 \rfloor = 7$ $\lceil 7 \rceil = 7$

Note: $x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$