Chapter 3

1

VECTORS

Vectors

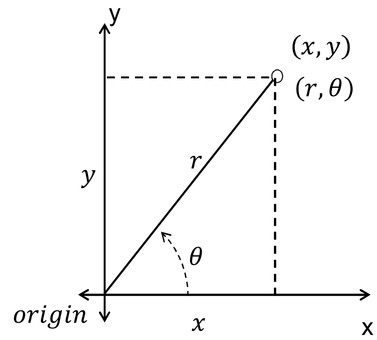
2

- Vector quantities
  - Physical quantities that have both numerical and directional properties.
- Mathematical operations of vectors in this chapter
  - Addition
  - Subtraction
  - Multiplication:
    - Multiplying with a scalar
    - Scalar product
    - Vector product

Coordinate Systems

3

- Used to describe the position of a point in space
- Common coordinate systems are:
  - Cartesian
  - Polar



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Polar to Cartesian Coordinates

4

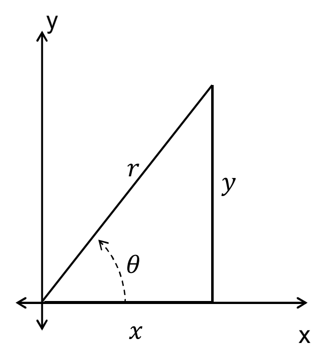
- Based on forming a right triangle from r and θ

$$cos\theta = \frac{x}{r} \Leftrightarrow x = rcos\theta$$

$$sin\theta = \frac{y}{r} \Leftrightarrow y = rsin\theta$$

- If the Cartesian coordinates are known:

$$r = \sqrt{x^2 + y^2}$$
$$tan\theta = \frac{y}{x} \Leftrightarrow \theta = tan^{-1}\frac{y}{x}$$



x: adjacent

y: opposite

r: hypotenuse

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Vectors and Scalars

5

A scalar quantity is completely specified by a magnitude with an appropriate unit and has no direction.

- Many are always positive
- Some may be positive or negative
- Rules for ordinary arithmetic are used to manipulate scalar quantities.

A vector quantity is completely described by a magnitude, an appropriate units and a direction.

Vector and scalar Examples

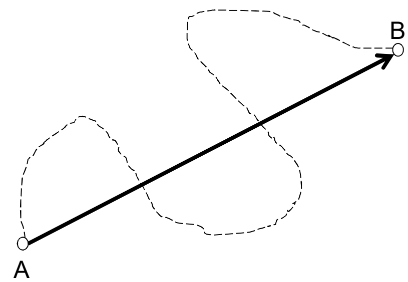
6

SCALARS	VECTORS
• Distance	• Displacement
• Speed	• Velocity
• Time intervals	• Acceleration
• Mass	• Force

Vector and scalar Example

7

- A particle travels from A to B along the path shown by the broken line.
  - This is the distance traveled and is a scalar.
- The displacement is the solid line from A to B
  - The displacement is independent of the path taken between the two points.
  - Displacement is a vector.



Vector Notation

8

Bold font is used for printing: **A**

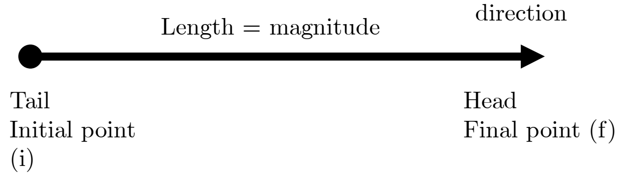
When dealing with just the magnitude of a vector in print, an italic letter will be used: A or $|\vec{A}|$

The magnitude of a vector is always a positive number.

When handwritten, use an arrow: $\vec{A}$

Vectors representation

9



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Vectors representation

10

$$\vec{A} = \left|\vec{A}\right|, \theta$$

Vector A

Its magnitude

Its direction

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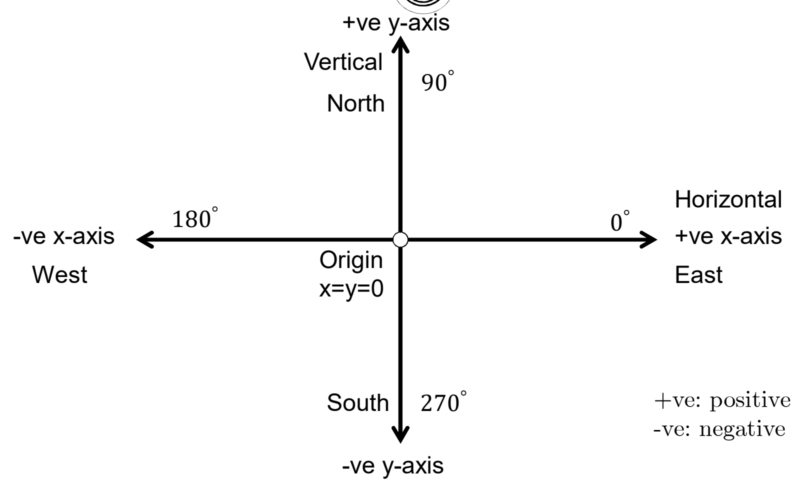
Directions

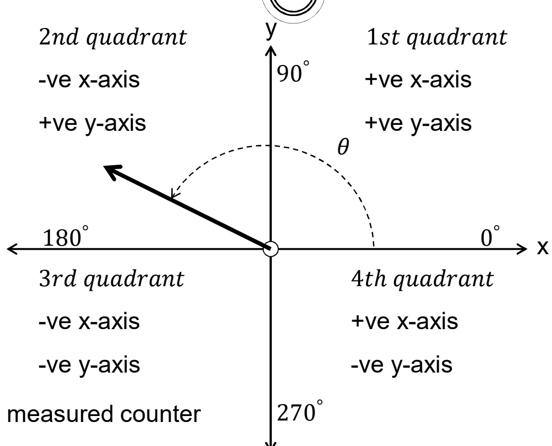
11



+ve: positive
-ve: negative

Directions ($\theta \equiv Theta$)

12



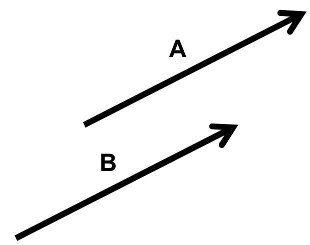
(θ) : should be measured counter clockwise starting at the positive x-axis

Equality of Two Vectors

13

- Two vectors are equal if:
  - they have the same magnitude and
  - points in the same direction.

- $\vec{A} = \vec{B}$ if:
- $|\vec{A}| = |\vec{B}|$ and
  - In the same direction, or
  - Parallel, or
  - $\theta_{A,B} = 0^{\circ}$



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Adding Vectors

14

- Vector addition is very different from adding scalar quantities.
- When adding vectors, their directions must be taken into account.
- Units must be the same
- Graphical methods
  - Use scale drawings
- Algebraic methods
  - More convenient

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Adding Vectors Graphically

15

- Choose a scale.
- Draw the first vector, $\vec{A}$, with the appropriate length and in the direction specified, with respect to a coordinate system.
- Draw the next vector with the appropriate length and in the direction specified, with respect to a coordinate system whose origin is the end of vector $\vec{A}$ and parallel to the coordinate system used for $\vec{A}$.
- Continue drawing the vectors “tip-to-tail” or “head-to-tail”.
- The resultant is drawn from the origin of the first vector to the end of the last vector.

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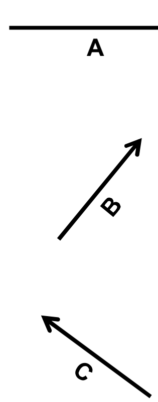
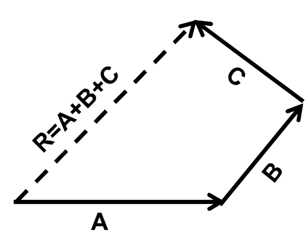
Adding Vectors Graphically, cont.

16





R: resultant

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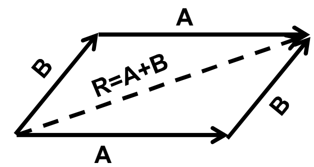
Adding Vectors, Rules

17

○The Commutative Law of Addition: when two vectors are added, the sum is independent of the order of the addition.

- This is

$$\vec{R} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$



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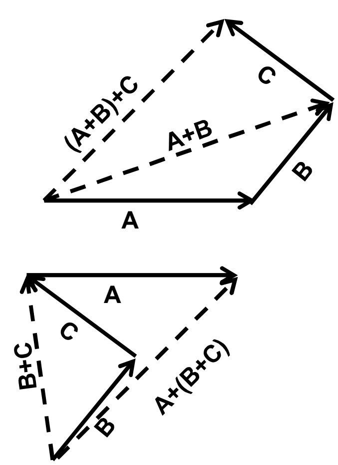
Adding Vectors, Rules cont.

18

•The Associative Property of Addition: when adding three or more vectors, their sum is independent of the way in which the individual vectors are grouped.

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$



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Adding Vectors, Rules final

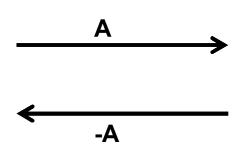
19

- When adding vectors, all of the vectors must have the same units.
- All of the vectors must be of the same type of quantity.
  - For example, you cannot add a displacement to a velocity.

Negative of a Vector

20

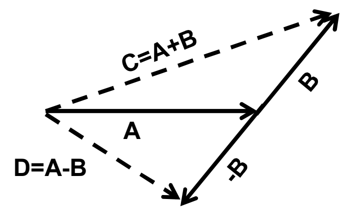
- The negative of a vector is defined as the vector that, when added to the original vector, gives a resultant of zero.
  - Represented as $-\vec{A}$
  - $\vec{A} + (-\vec{A}) = 0$
- The negative of the vector will have the same magnitude, but points in the opposite direction.



Subtracting Vectors

21

- Special case of vector addition.
- If $\vec{A} - \vec{B}$, then use $\vec{A} + (-\vec{B})$
- Continue with standard vector addition procedure.



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Subtracting Vectors, Method 2

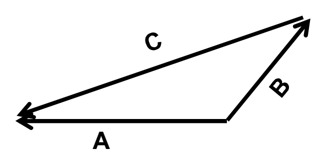
22

In the figure shown:

$$\vec{B} + \vec{C} = \vec{A}$$

$$\vec{C} = \vec{A} - \vec{B}$$

- The resultant vector $\vec{C}$ of subtracting $\vec{B}$ from $\vec{A}$ points from the tip of the second to the tip of the first.



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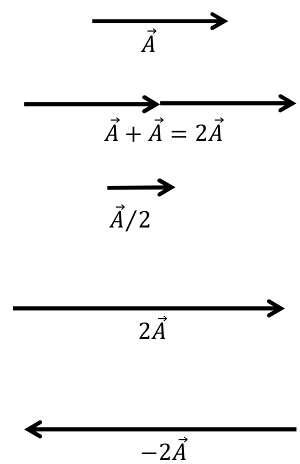
Multiplying or Dividing a Vector by a Scalar

- The result of the multiplication or division of a vector by a scalar is a vector.
- The magnitude of the vector is multiplied or divided by the scalar.
- If the scalar is positive, the direction of the result is the same as of the original vector.
- If the scalar is negative, the direction of the result is opposite that of the original vector.



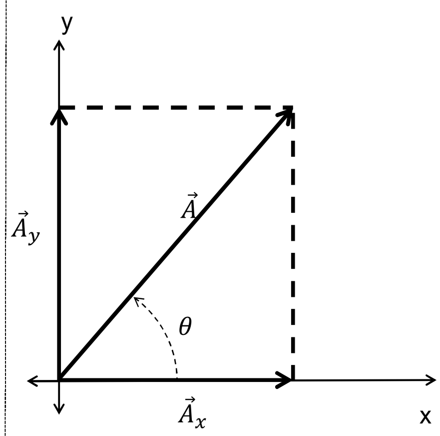
Component Method of Adding Vectors

- Graphical addition is not recommended when:
  - High accuracy is required
  - If you have a three-dimensional problem
- Component method is an alternative method
  - It uses projections of vectors along coordinate axes

Components of a Vector, Introduction

25

- A **component** is a projection of a vector along an axis.
  - A_x : the component (projection) of the vector along the x-axes.
  - A_y : the component (projection) of the vector along the y-axes.



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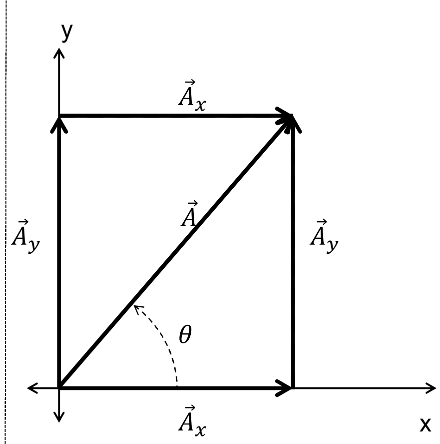
Components of a Vector, Introduction

26

$$\vec{A}_x + \vec{A}_y = \vec{A}$$

$$\vec{A}_y + \vec{A}_x = \vec{A}$$



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Vector Component Terminology

27

- $\vec{A}_x$ and $\vec{A}_y$ are the component vectors of $\vec{A}$.
  - They are vectors and follow all the rules for vectors.
- $|\vec{A}_x|$ and $|\vec{A}_y|$ are scalars, and will be referred to as the components of $\vec{A}$.

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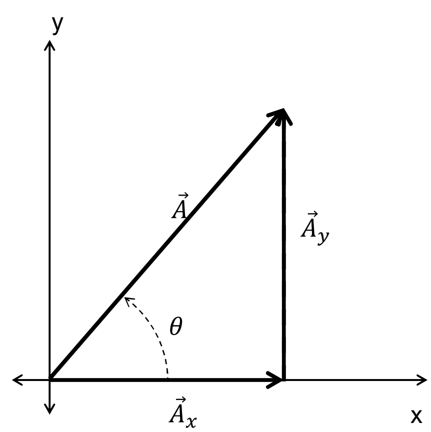
Components of a Vector

28

- Assume you are given a vector $\vec{A}$
- It can be expressed in terms of two other vectors, $\vec{A}_x$ and $\vec{A}_y$

$$\vec{A}_x + \vec{A}_y = \vec{A}$$

- These three vectors form a right triangle.



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Components of a Vector

29

The x-component is:

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{A_x}{A}$$

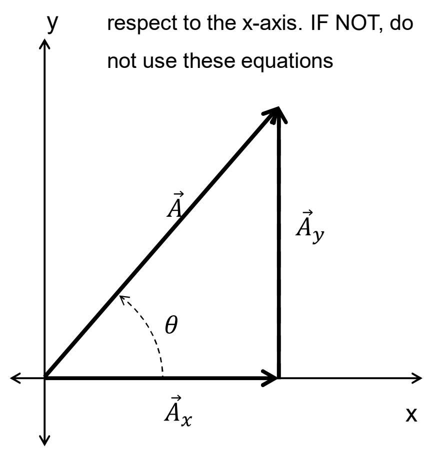
$$A_x = A\cos\theta$$

The y-component is:

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{A_y}{A}$$

$$A_y = A\sin\theta$$

The angle θ is measured with respect to the x-axis. IF NOT, do not use these equations



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Components of a Vector

30

The magnitude of the vector is:

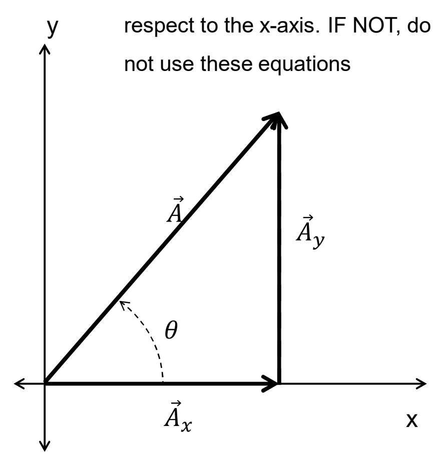
$$A = \sqrt{A_x^2 + A_y^2}$$

The direction of the vector is:

$$\tan\theta = \frac{opposite}{adjacent} = \frac{A_y}{A_x}$$

$$\theta = \tan^{-1}\frac{A_y}{A_x}$$

The angle θ is measured with respect to the x-axis. IF NOT, do not use these equations



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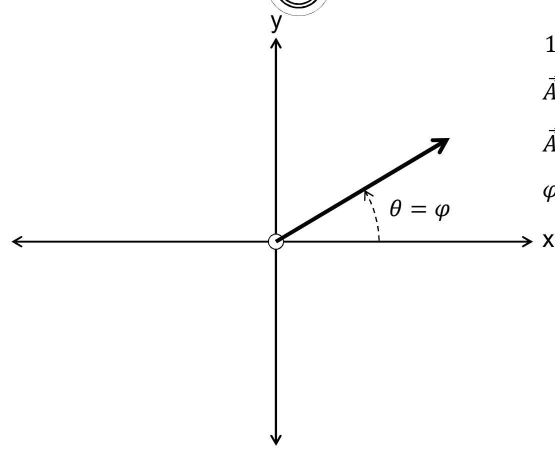
Direction (θ) and calculations from your calculator (φ)

31



1st quadrant

$\vec{A}_x$: +ve

$\vec{A}_y$: +ve

φ : +ve

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Direction (θ) and calculations from your calculator (φ)

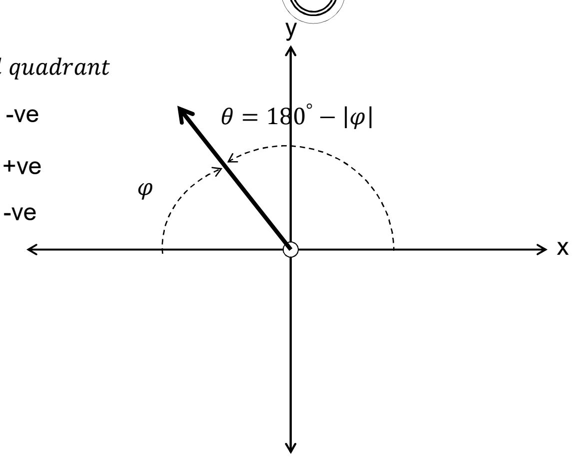
32

2nd quadrant

$\vec{A}_x$: -ve

$\vec{A}_y$: +ve

φ : -ve



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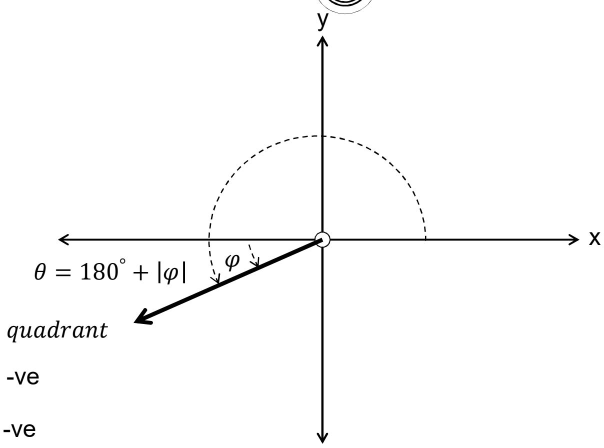
Direction (θ) and calculations from your calculator (φ)



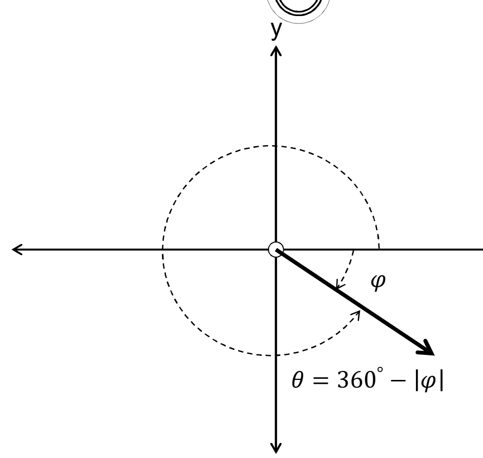
3rd quadrant

$\vec{A}_x$: -ve

$\vec{A}_y$: -ve

φ : +ve

Direction (θ) and calculations from your calculator (φ)



4th quadrant

$\vec{A}_x$: +ve

$\vec{A}_y$: -ve

φ : -ve

Unit Vectors

- A unit vector is a dimensionless vector with a magnitude of exactly 1.
- Unit vectors are used to specify a direction and have no other physical significance.

Unit Vectors, cont.

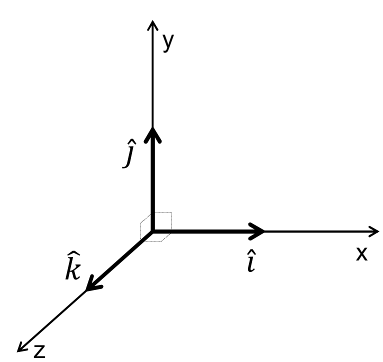
- The symbols $\hat{\imath}, \hat{\jmath}$ *and* $\hat{k}$ represent unit vectors.
- The magnitude of each unit vector is 1.

$$\hat{\imath} = \hat{\jmath} = \hat{k} = 1$$

- They form a set of mutually perpendicular vectors in a right-handed coordinate system.

$$\hat{\imath} \perp \hat{\jmath} \perp \hat{k}$$

- Dimensionless vectors.



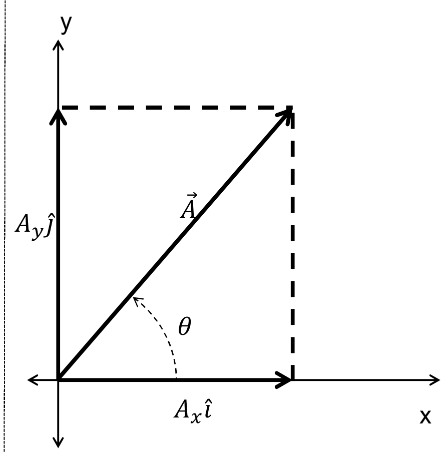
Unit Vectors in Vector Notation

- $\vec{A}_x$ is the same as $A_x\hat{\imath}$, $\vec{A}_y$ is the same as $A_y\hat{\jmath}$ and $\vec{A}_z$ is the same as $A_z\hat{k}$.
- The complete vector can be expressed as:

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

$$\vec{A} = Acos\theta\hat{\imath} + Asin\theta\hat{\jmath}$$



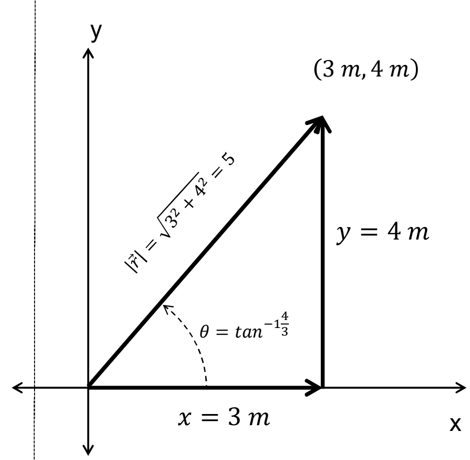
Position Vector, Example

- If $x = 3m$ and $y = 4m$:

$\vec{r} = 3 + 4 = 7$ X

$\vec{r} = \vec{3}_x + \vec{4}_y$?

$\vec{r} = 3\hat{\imath} + 4\hat{\jmath}$ √



Adding Vectors Using Unit Vectors

39

- Given:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

$$\downarrow \quad \downarrow$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$$

- Find: $\vec{R} = \vec{A}+\vec{B}$

$$\vec{R} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath}$$

$$\vec{R} = R_x\hat{\imath} + R_y\hat{\jmath}$$

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Adding Vectors Using Unit Vectors

40

So, the x-component of the resultant is:

$$R_x = A_x + B_x$$

The y-component of the resultant is:

$$R_y = A_y + B_y$$

The magnitude of the resultant is:

$$R = \sqrt{R_x^2 + R_y^2}$$

The direction of the resultant is:

$$\theta = tan^{-1}\frac{R_y}{R_x}$$

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Three-Dimensional Extension

41

- Given:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

- Find: $\vec{R} = \vec{A}+\vec{B}$

$$\vec{R} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}$$
$$\vec{R} = R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k}$$

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Three-Dimensional Extension

42

So, the x-component of the resultant is:

$$R_x = A_x + B_x$$

The y-component of the resultant is:

$$R_y = A_y + B_y$$

The z-component of the resultant is:

$$R_z = A_z + B_z$$

The magnitude of the resultant is:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

The direction of the resultant is:

$LATER$

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Adding Three or More Vectors

43

- The same method can be extended to adding three or more vectors.
- Assume

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\vec{C} = C_x\hat{\imath} + C_y\hat{\jmath} + C_z\hat{k}$$

- And

$$\vec{R} = \vec{A}+\vec{B}+\vec{C}$$

$$\vec{R} = (A_x + B_x + C_x)\hat{\imath} + (A_y + B_y + C_y)\hat{\jmath} + (A_z + B_z + C_z)\hat{k}$$

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Scalar Product of Two Vectors

44

The scalar product of two vectors is defined as.

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

- It is also called the dot product.

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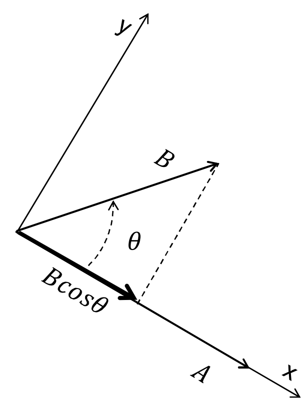
Scalar Product of Two Vectors

45

- θ is the angle between $\vec{A}$ and $\vec{B}$ when they are drawn starting at the same point.
- The scalar product of any two vectors is a scalar quantity.



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Scalar Product, cont

46

The scalar product is commutative.

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

The scalar product obeys the distributive law of multiplication.

$$\vec{A} \cdot (\vec{B} + \vec{C}) = (\vec{A} \cdot \vec{B}) + (\vec{A} \cdot \vec{C})$$

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Dot Products of Unit Vectors

$$\hat{\imath} \cdot \hat{\imath} = (1)(1)cos0^{\circ} = 1$$

$$\hat{\imath} \cdot \hat{\jmath} = (1)(1)cos90^{\circ} = 0$$

- In general:

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$$

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\imath} \cdot \hat{k} = \hat{\jmath} \cdot \hat{k} = 0$$

- Using component form with vectors:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\Rightarrow \quad \vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Dot Products of Unit Vectors

Special cases:

$$\vec{A} \cdot \vec{B} = \begin{cases} AB; If\ they\ are\ parallel\ ;\ \theta = 0^{\circ} \\ -AB\ ;\ If\ they\ are\ anti-parallel\ ;\ \theta = 180^{\circ} \\ 0\ ; If\ they\ are\ perpendicular\ ;\ \theta = 90^{\circ} \end{cases}$$

$$\vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2 = A^2$$

The Vector Product

- There are instances where the product of two vectors is another vector.
  - Earlier we saw where the product of two vectors was a scalar.
- The vector product of two vectors is called the cross product.

The Vector Product Defined

The vector (cross) product of $\vec{A}$ and $\vec{B}$ is defined as a third vector:

$$\vec{C} = \vec{A} \times \vec{B}$$

The magnitude of $\vec{C}$:

$$|\vec{C}| = |\vec{A}||\vec{B}|sin\theta$$

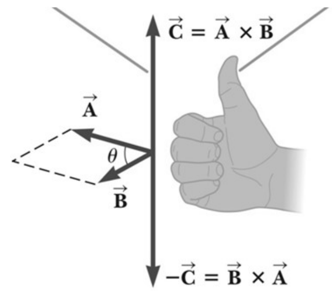
θ is the angle between $\vec{A}$ and $\vec{B}$ when they are drawn starting at the same point.

More About the Vector Product

51

The direction of $\vec{C}$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$.

The best way to determine this direction is to use the right-hand rule (R.H.R).



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Properties of the Vector Product

52

The vector product is not commutative. The order in which the vectors are multiplied is important.

To account for order, remember:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

If $\vec{A}$ is parallel to $\vec{B}$ ($\theta = 0^\circ \; or 180^\circ$), then

$$\vec{A} \times \vec{B} = 0$$

Therefore

$$\vec{A} \times \vec{A} = 0$$

If $\vec{A}$ is perpendicular to $\vec{B}$, then

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|$$

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Properties of the Vector Product

53

The vector product obeys the distributive law:

$$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$$

The derivative of the cross product with respect to some variable such as t is:

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \left(\frac{d\vec{A}}{dt} \times \vec{B}\right) + \left(\vec{A} \times \frac{d\vec{B}}{dt}\right)$$

where it is important to preserve the multiplicative order of the vectors.

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Vector Products of Unit Vectors

54

$\hat{\imath} \times \hat{\imath} = (1)(1)sin0^{\circ} = 0$

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$

$\hat{\imath} \times \hat{\jmath} = \hat{k}$

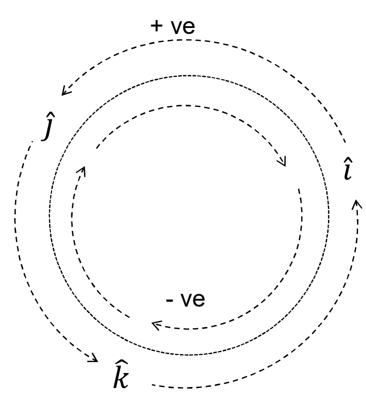
$\hat{\jmath} \times \hat{k} = \hat{\imath}$

$\hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{\imath} = -\hat{k}$

$\hat{k} \times \hat{\jmath} = -\hat{\imath}$

$\hat{\imath} \times \hat{k} = -\hat{\jmath}$



26-Oct-20

Signs in Cross Products

55

- Signs are interchangeable in cross products

$$\vec{A} \times (-\vec{B}) = -\vec{A} \times \vec{B}$$

Using Determinants

56

- The cross product can be expressed as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = +(A_y B_z - A_z B_y)\hat{\imath} - (A_x B_z - A_z B_x)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k}$$