# Chapter 9

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## Linear Momentum and Collisions



## Momentum Analysis Models

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- Force and acceleration are related by Newton's second law.
- When force and acceleration vary by time, the situation can be very complicated.
- The techniques developed in this chapter will enable you to understand and analyze these situations in a simple way.
- Will develop momentum versions of analysis models for isolated and non-isolated systems
- These models are especially useful for treating problems that involve collisions and for analyzing rocket propulsion.

## Thought Experiment

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- An archer stands on frictionless ice and fires an arrow. What is the archer's velocity after firing the arrow?
  - Motion models such as a particle under constant acceleration cannot be used.
    - No information about the acceleration of the arrow
  - Model of a particle under constant force cannot be used.
    - No information about forces involved
  - Energy models cannot be used.
    - No information about the work or the energy (energies) involved
- A new quantity is needed – linear momentum.



## Linear Momentum

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Consider newton's second law:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt}$$

With constant mass.

The net force is equal to the change in the product $(m\vec{v})$ per unit time.

This product is called the linear momentum (or the momentum).

# Linear Momentum

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The linear momentum of a particle or an object of mass $m$ moving with a velocity $\vec{v}$ is defined to be the product of the mass and velocity:

$$\vec{p} = m\vec{v}$$

Linear momentum is a vector quantity.

Its direction is the same as the direction of the velocity.

The SI units of momentum are:

$$kg.m/s$$

Momentum can be expressed in component form:

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z$$



# Conservation of Linear Momentum

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Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.

- The momentum of the system is conserved, not necessarily the momentum of an individual particle.
  - Avoid applying conservation of momentum to a single particle.
- This also tells us that the total momentum of an isolated system equals its initial momentum.

# Conservation of Momentum

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Conservation of momentum can be expressed mathematically in various ways:

$$\vec{p}_{total} = \vec{p}_1 + \vec{p}_2 = Constant$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

In component form, the total momenta in each direction are independently conserved.

$$\vec{p}_{1ix} + \vec{p}_{2i} = \vec{p}_{1fx} + \vec{p}_{2fx}$$

$$\vec{p}_{1iy} + \vec{p}_{2iy} = \vec{p}_{1fy} + \vec{p}_{2fy}$$

$$\vec{p}_{1iz} + \vec{p}_{2iz} = \vec{p}_{1fz} + \vec{p}_{2fz}$$

Conservation of momentum can be applied to systems with any number of particles.



# Forces and Conservation of Momentum

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In conservation of momentum, there is no statement concerning the types of forces acting on the particles of the system.

The forces are not specified as conservative or non-conservative.

There is no indication if the forces are constant or not.

The only requirement is that the forces must be internal to the system.

- This gives a hint about the power of this new model.

# Newton's Second Law and Momentum

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Newton's Second Law:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

The time rate of change of the linear momentum of a particle is equal to the net force acting on the particle.

- This is the form in which Newton presented the Second Law.
- It is a more general form than the one we used previously.
- This form also allows for mass changes.



# Impulse and Momentum

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The momentum of a system changes if a net force from the environment acts on the system.

For momentum considerations, a system is non-isolated if a net force acts on the system for a time interval.

## Impulse and Momentum

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- From Newton's Second Law, $\vec{F} = \frac{d\vec{p}}{dt}$
- Solving for $d\vec{p}$ gives $d\vec{p} = \vec{F}dt$
- Integrating to find the change in momentum over some time interval.

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} \vec{F}dt$$

- The integral is called the *impulse*, $\vec{I}$ , of the force acting on an object over $\Delta t$.



## Impulse-Momentum Theorem

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The change in the momentum of a particle is equal to the impulse of the new force acting on the particle.

$$\Delta\vec{p} = \vec{I}$$

- This is equivalent to Newton's Second Law.
- This is identical in form to the conservation of energy equation.
- This is the most general statement of the principle of conservation of momentum and is called the conservation of momentum equation.
  - This form applies to non-isolated systems.
- This is the mathematical statement of the non-isolated system (momentum) model.

# More About Impulse

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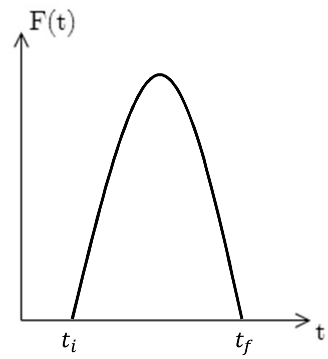
- Impulse is a vector quantity.
- The magnitude of the impulse is equal to the area under the force-time curve.
    - The force may vary with time.
- Impulse is not a property of the particle, but a measure of the change in momentum of the particle.






# Impulse, Final

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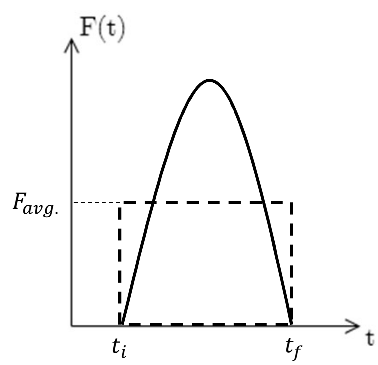
- The impulse can also be found by using the time averaged force.

$$\vec{I} = \vec{F}_{avg.}\Delta t$$

- This would give the same impulse as the time-varying force does.



# Impulse Approximation

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- In many cases, one force acting on a particle acts for a short time, but is much greater than any other force present.
- When using the Impulse Approximation, we will assume this is true.
  - Especially useful in analyzing collisions
- The force will be called the *impulsive force.*
- The particle is assumed to move very little during the collision.
- $\vec{p}_i$ and $\vec{p}_f$ represent the momenta *immediately* before and after the collision.



# Momentum and Kinetic Energy

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- Momentum and kinetic energy both involve mass and velocity.
- There are major differences between them:
  - Kinetic energy is a scalar and momentum is a vector.
  - Kinetic energy can be transformed to other types of energy.
    - There is only one type of linear momentum, so there are no similar transformations.
- Analysis models based on momentum are separate from those based on energy.
- This difference allows an independent tool to use in solving problems.

# Collisions – Characteristics

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The term collision represents an event during which two particles come close to each other and interact by means of forces.

- May involve physical contact, but must be generalized to include cases with interaction without physical contact

The interaction forces are assumed to be much greater than any external forces present.

- This means the impulse approximation can be used.



# Collisions – Example 1

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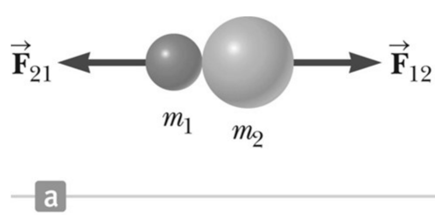
- Collisions may be the result of direct contact.
- The impulsive forces may vary in time in complicated ways.
  - This force is internal to the system.
- Momentum is conserved.



# Collisions – Example 2

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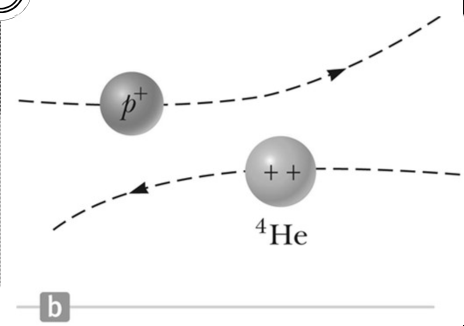
- The collision need not include physical contact between the objects.
- There are still forces between the particles.
- This type of collision can be analyzed in the same way as those that include physical contact.






# Types of Collisions

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- Elastic collision
  - Both momentum and kinetic energy are conserved
- Inelastic collision
  - kinetic energy is not conserved
  - momentum is still conserved.
- Perfectly inelastic collision
  - kinetic energy is not conserved
  - momentum is still conserved.
  - the objects stick together after the collision

# Types of Collisions

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Elastic collisions occur on a microscopic level.

- In macroscopic collisions, only approximately elastic collisions actually occur.
- Generally some energy is lost to deformation, sound, etc.

In an inelastic collision, some kinetic energy is lost, but the objects do not stick together.

Elastic and perfectly inelastic collisions are limiting cases, most actual collisions fall in between these two types .

Momentum is conserved in all collisions



# Perfectly Inelastic Collisions

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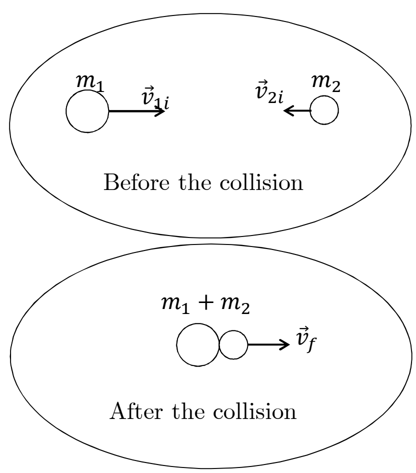
- Momentum of an isolated system is conserved in any collision, so the total momentum before the collision is equal to the total momentum of the composite system after the collision.
- Since the objects stick together, they share the same velocity after the collision.

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$



# Elastic Collisions

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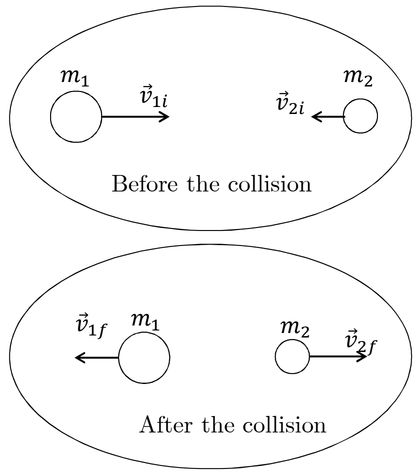
- Both momentum and kinetic energy are conserved.

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

- Typically, there are two unknowns to solve for and so you need two equations.






# Elastic Collisions

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Equations of conservation of linear momentum and conservation of kinetic energy can be solved for the final velocities in terms of the initial velocities and masses:

$$\vec{v}_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{v}_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)\vec{v}_{2i}$$

$$\vec{v}_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)\vec{v}_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)\vec{v}_{2i}$$

Can only be used with a one-dimensional, elastic collision.

Remember to use the appropriate signs for all velocities.

# Elastic Collisions, Some special cases

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$m_1 = m_2$ ; the particles exchange velocities.

- $v_{1f} = v_{2i}$
- $v_{2f} = v_{1i}$

When a very heavy particle $(m_1)$ collides head-on with a very light one $(m_2)$ initially at rest $(m_1 \gg m_2$ and $v_{2i} = 0)$, the heavy particle continues in motion unaltered and the light particle rebounds with a speed of about twice the initial speed of the heavy particle.

- $v_{1f} \cong v_{1i}$
- $v_{2f} \cong -2v_{2i}$

When a very light particle $(m_1)$ collides head-on with a very heavy particle $(m_2)$ initially at rest $(m_1 \ll m_2$ and $v_{2i} = 0)$, the light particle has its velocity reversed and the heavy particle remains approximately at rest.

- $v_{1f} = -v_{1i}$
- $v_{2f} \cong 0$



# Two-Dimensional Collisions

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- Same as in 1-D.
- Vector Momentum Conservation:

$$m_1\vec{v}_{1xi} + m_2\vec{v}_{2x} = m_1\vec{v}_{1xf} + m_2\vec{v}_{2xf}$$

$$m_1\vec{v}_{1yi} + m_2\vec{v}_{2yi} = m_1\vec{v}_{1yf} + m_2\vec{v}_{2yf}$$

- If the collision is elastic, Kinetic Energy Conservation:

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

  - The simpler equations can only be used for one-dimensional situations.

## Two-Dimensional Collision, example

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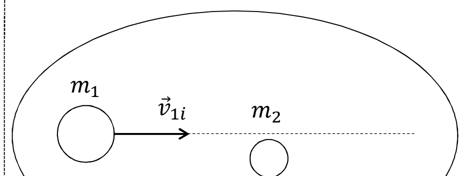
Particle 1 is moving at velocity $\vec{v}_{1i}$ and particle 2 is at rest.

Before the collision:

- The initial momentum in the $x$-direction, is:

$$m_1\vec{v}_{1i}$$

- The initial momentum in the $y$-direction, is 0.




Before the collision



## Two-Dimensional Collision, example cont.

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After the collision:

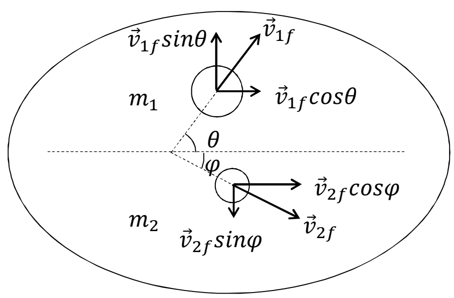
- The momentum in the $x$-direction is:

$$m_1\vec{v}_{1f}cos\theta + m_2\vec{v}_{2f}cos\varphi$$

- After the collision, the momentum in the $y$-direction is

$$m_1\vec{v}_{1f}sin\theta - m_2\vec{v}_{2f}sin\varphi$$

  - The negative sign is due to the component of the velocity being downward.
- If the collision is elastic, apply the kinetic energy equation.




After the collision

# The Center of Mass

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There is a special point in a system or object, called the center of mass, that moves as if all of the mass of the system is concentrated at that point.

The system will move as if an external force were applied to a single particle of mass M located at the center of mass.

M is the total mass of the system.

This behavior is independent of other motion, such as rotation or vibration, or deformation of the system.

This is the particle model.



# Center of Mass, Coordinates

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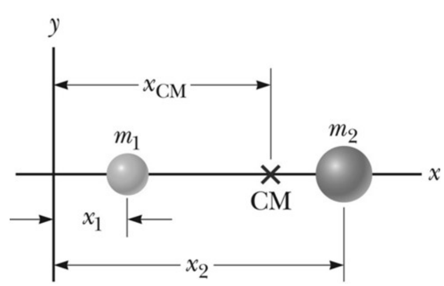
The coordinates of the center of mass are

$$X_{com} = \frac{\sum_i m_i x_i}{M} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots}$$

$$Y_{com} = \frac{\sum_i m_i y_i}{M} = \frac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 + m_2 + \cdots}$$

$$Z_{com} = \frac{\sum_i m_i z_i}{M} = \frac{m_1 z_1 + m_2 z_2 + \cdots}{m_1 + m_2 + \cdots}$$

M is the total mass of the system.



# Center of Mass, position

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For a system of particles, the center of mass in three dimensions can be located by its position vector, $\vec{r}_{com}$ .

- $\vec{r}_{com} = X_{com}\hat{\imath} + Y_{com}\hat{\jmath} + Z_{com}\hat{k}$



# Motion of a System of Particles

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- Assume the total mass, M, of the system remains constant.
- We can describe the motion of the system in terms of the velocity and acceleration of the center of mass of the system.
- We can also describe the momentum of the system and Newton's Second Law for the system.

## Velocity and Momentum of a System of Particles

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The velocity of the center of mass of a system of particles is:

- $\vec{v}_{com} = \frac{\sum_i m_i \vec{v}_i}{M} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots}{m_1 + m_2 + \cdots}$

The momentum can be expressed as:

- $M\vec{v}_{com} = \sum_i m_i \vec{v}_i = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots = \vec{p}_{tot}$

The total linear momentum of the system equals the total mass multiplied by the velocity of the center of mass.



## Acceleration and Force in a System of Particles

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The acceleration of the center of mass can be found by differentiating the velocity with respect to time.

- $\vec{a}_{com} = \frac{\sum_i m_i \vec{a}_i}{M} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots}{m_1 + m_2 + \cdots}$

The acceleration can be related to a force.

- $M\vec{a}_{com} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots = \vec{F}_1 + \vec{F}_2 + \cdots$

If we sum over all the internal force vectors, they cancel in pairs and the net force on the system is caused only by the external forces.

## Newton's Second Law for a System of Particles

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•Since the only forces are external, the net external force equals the total mass of the system multiplied by the acceleration of the center of mass:

$$\sum \vec{F}_{ext} = M\vec{a}_{com}$$

•The center of mass of a system of particles of combined mass *M* moves like an equivalent particle of mass *M* would move under the influence of the net external force on the system.



## Impulse and Momentum of a System of Particles

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•The impulse imparted to the system by external forces is

$$\int \sum \vec{F}_{ext}\, dt = M \int d\vec{v}_{com}$$

$$\Delta\vec{p}_{tot} = \vec{I}$$

•The total linear momentum of a system of particles is conserved if no net external force is acting on the system.

$$M\vec{v}_{com} = \vec{p}_{total} = constant\ when \sum \vec{F}_{ext} = 0$$

•For an isolated system of particles, both the total momentum and the velocity of the center of mass are constant in time.

- This is a generalization of the isolated system (momentum) model for a many-particle system.