Chapter 10

1

Rotation Of A Rigid Object
About A Fixed Axis

Rigid Object

2

Analysis models introduced so far cannot be used to analyze all motion.

We can model the motion of an extended object by modeling it as a system of many particles.

- The analysis is simplified if the object is assumed to be a **rigid object.**

A rigid object is one that is non-deformable.

- The relative locations of all particles making up the object remain constant.
- All real objects are deformable to some extent, but the rigid object model is very useful in many situations where the deformation is negligible.

In this chapter another class of analysis models based on the rigid-object model are developed.

Angular Position

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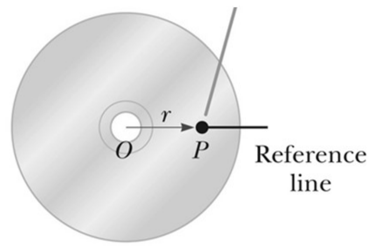
- Axis of rotation is the center of the disc
- Choose a fixed reference line.
- Point P is at a fixed distance r from the origin.
  - A small element of the disc can be modeled as a particle at P.



Angular Position

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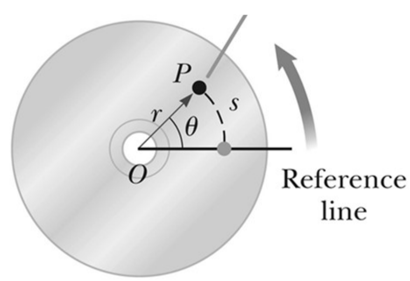
- Polar coordinates are convenient to use to represent the position of P (or any other point).
- P is located at (r, θ) where r is the distance from the origin to P and θ is the measured counterclockwise from the reference line.



Angular Position, cont.

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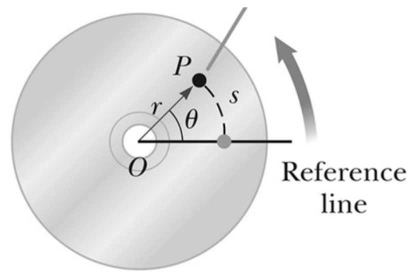
- As the particle moves, the only coordinate that changes is θ.
- As the particle moves through θ, it moves though an arc length s.
- The arc length (s) and (r) are related:

$$\boxed{s = r\theta}$$



Radian

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This can also be expressed as:

$$\theta = \frac{s}{r}$$

- θ is a pure number, but commonly is given the artificial unit, radian.
- One radian is the angle subtended by an arc length equal to the radius of the arc.
- Whenever using rotational equations, you must use angles expressed in radians.

Conversions

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Comparing degrees and radians	• $1\ rad = \frac{360^{\circ}}{2\pi} = 57.3^{\circ}$
Converting from degrees to radians	• $\theta\ (rad) = \frac{\pi}{180^{\circ}}\theta(degrees)$

Angular Position, final

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- We can associate the angle θ with the entire rigid object as well as with an individual particle.
  - Remember every particle on the object rotates through the same angle.
- The **angular position** of the rigid object is the angle θ between the reference line on the object and the fixed reference line in space.
  - The fixed reference line in space is often the x-axis.
- The angle θ plays the same role in rotational motion that the position x does in translational motion.

Angular Displacement

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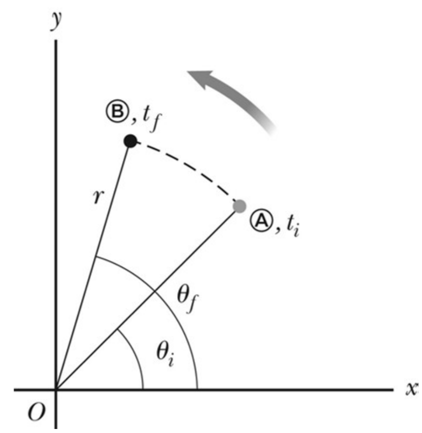
The *angular displacement* is defined as the angle the object rotates through during some time interval.

$$\Delta\theta = \theta_f - \theta_i$$

- This is the angle that the reference line of length r sweeps out.



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Average Angular Speed

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The *average* angular speed, ω_{avg}, of a rotating rigid object is the ratio of the angular displacement to the time interval.

- $\omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$

The *instantaneous* angular speed is defined as the limit of the average speed as the time interval approaches zero.

- $\omega = \frac{d\theta}{dt}$

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Angular Speed

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- This is analogous to translational speed.
- Units of angular speed are **radians/sec**.
  - (rad/s or s^{-1} since radians have no dimensions).
- Angular speed will be positive if θ is increasing (counterclockwise)
- Angular speed will be negative if θ is decreasing (clockwise)

Angular Acceleration

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The average angular acceleration, α_{avg} , of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change.

- $\alpha_{avg} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$

The instantaneous angular acceleration is defined as the limit of the average angular acceleration as the time goes to 0.

- $\alpha = \frac{d\omega}{dt}$

Angular Acceleration, cont.

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- Analogous to translational velocity
- Units of angular acceleration are $\boldsymbol{rad/s^2}$ **or** $\boldsymbol{s^{-2}}$ since radians have no dimensions.
- Angular acceleration will be positive if an object rotating counterclockwise is speeding up.
- Angular acceleration will also be positive if an object rotating clockwise is slowing down.

Angular Motion, General Notes

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- When a rigid object rotates about a fixed axis in a given time interval, every portion on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration.
  - So: $(\theta, \omega \text{ } and \text{ } \alpha)$ all characterize the motion of the entire rigid object as well as the individual particles in the object.

Directions, details

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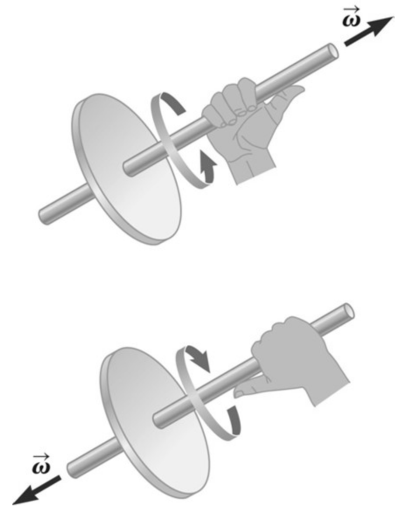
- Strictly speaking, the speed and acceleration (ω *and* α) are the magnitudes of the velocity and acceleration vectors.
- The directions are actually given by the right-hand rule.



Rotational Kinematics

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- Under constant angular acceleration, we can describe the motion of the rigid object using a set of kinematic equations.
  - These are similar to the kinematic equations for linear motion.
  - The rotational equations have the same mathematical form as the linear equations.
- The new model is a rigid object under constant angular acceleration.
  - Analogous to the particle under constant acceleration model.

Rotational Kinematic Equations

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Equations	Missing
$\omega_f = \omega_i + \alpha t$	$\Delta\theta$: displacement (m)
$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$	t : time (s)
$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$	ω_f : final velocity (m/s)
$\Delta\theta = \omega_f t - \frac{1}{2}\alpha t^2$	ω_i : initial velocity (m/s)
$\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)t$	α : acceleration (m/s^2)

Relationship Between Angular and Linear Quantities

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Every point on the rotating object has the same angular motion.

Every point on the rotating object does **not** have the same linear motion.

Displacements	• $s = r\theta$
Speeds	• $v = r\omega$
Accelerations	• $a = r\alpha$

Speed Comparison – Details

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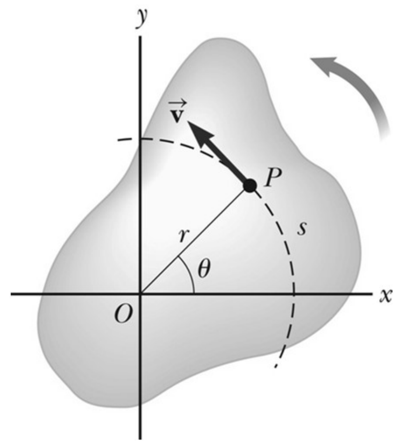
- The linear (tangential) velocity is always tangent to the circular path.
- The magnitude is defined by the tangential speed.

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$

- Since r is not the same for all points on the object, the tangential speed of every point is not the same.
- The tangential speed increases as one moves outward from the center of rotation.



Acceleration Comparison – Details

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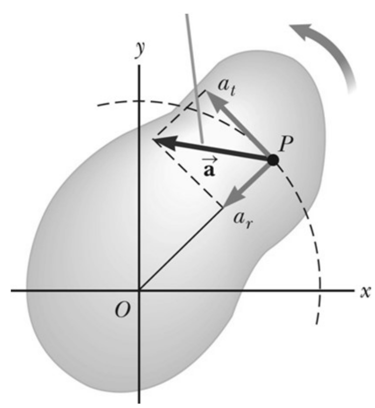
- The tangential acceleration is the derivative of the tangential velocity.

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$



Speed and Acceleration Note

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- All points on the rigid object will have the same angular speed, but not the same tangential speed.
- All points on the rigid object will have the same angular acceleration, but not the same tangential acceleration.
- The tangential quantities depend on r, and r is not the same for all points on the object.

Centripetal Acceleration

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An object traveling in a circle, even though it moves with a constant speed, will have an acceleration.

Therefore, each point on a rotating rigid object will experience a centripetal acceleration.

- $a_c = \frac{v^2}{r} = r\omega^2$

Resultant Acceleration

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The tangential component of the acceleration is due to changing speed.

The centripetal component of the acceleration is due to changing direction.

Total acceleration can be found from these components:

- $a = \sqrt{a_t^2 + a_c^2} = r\sqrt{\alpha^2 + \omega^4}$

Rotational Kinetic Energy

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An object rotating about some axis with an angular speed, ω, has rotational kinetic energy even though it may not have any translational kinetic energy.

Each particle has a kinetic energy of

$$K = \frac{1}{2} m_i v_i^2$$

Since the tangential velocity depends on the distance, r, from the axis of rotation, we can substitute

$$v_i = r\omega_i$$

Rotational Kinetic Energy, cont

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- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles.
- $K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} \omega^2 \sum_i m_i r_i^2$

$$K_R = \frac{1}{2} I \omega^2$$

- I is called the moment of inertia.

Rotational Kinetic Energy, final

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- There is an analogy between the kinetic energies associated with linear motion ($K = \frac{1}{2} m_i v_i^2$) and the kinetic energy associated with rotational motion ($K_R = \frac{1}{2} I \omega^2$).
- Rotational kinetic energy is not a new type of energy, the form is different because it is applied to a rotating object.
- The units of rotational kinetic energy are **Joules (J)**.

Moment of Inertia

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The definition of moment of inertia is

$$I = \sum_i m_i r_i^2$$

- The SI units are: $kg.m^2$
- Mass is an inherent property of an object, but the moment of inertia depends on the choice of rotational axis.
- Moment of inertia is a measure of the resistance of an object to changes in its rotational motion.
- The moment of inertia depends on the mass and how the mass is distributed around the rotational axis.

Moment of Inertia

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- The moment of inertia of a system of discrete particles can be calculated by applying the definition for I.
- For a continuous rigid object, imagine the object to be divided into many small elements, each having a mass of Δm_i.
- We can rewrite the expression for I in terms of Δm.

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

Moment of Inertia

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With the small volume segment assumption.

$$I = \int r^2 dm = \int \rho r^2 dV$$

- If r is constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known.

Notes on Various Densities

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- Volumetric Mass Density → mass per unit volume:

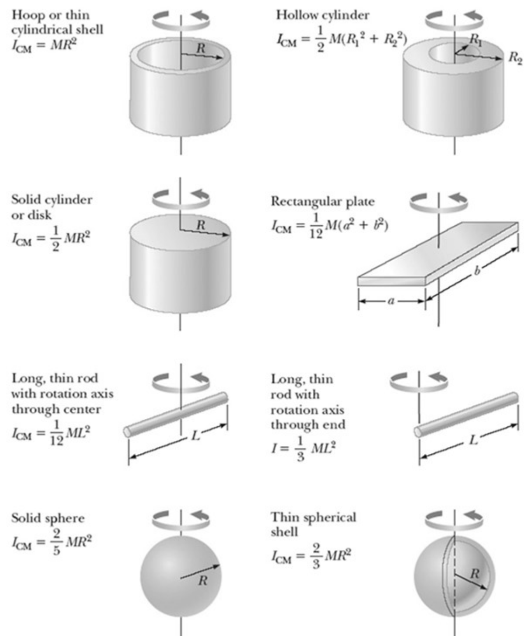
$$\rho = \frac{m}{V}$$

- Surface Mass Density → mass per unit thickness of a sheet of uniform thickness, t :

$$\sigma = \frac{m}{A} = \rho t$$

- Linear Mass Density → mass per unit length of a rod of uniform cross-sectional area:

$$\lambda = \frac{m}{\ell} = \rho A$$



Parallel-Axis Theorem

In the previous examples, the axis of rotation coincided with the axis of symmetry of the object.

For an arbitrary axis, the parallel-axis theorem often simplifies calculations.

The theorem states

$$I = I_{com} + MD^2$$

- I is about any axis parallel to the axis through the center of mass of the object.
- I_{com} is about the axis through the center of mass.
- D is the distance from the center of mass axis to the arbitrary axis.

Torque

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Torque,τ, is the tendency of a force to rotate an object about some axis.

Torque is a vector, but we will deal with its magnitude here:

$$\tau = |\vec{r}||\vec{F}|sin\varphi$$

F Is the force

Φ is the angle the force makes with the horizontal

r is the distance between the rotation axis and the point of application of $\vec{F}$.

There is no unique value of the torque on an object.

Its value depends on the choice of a rotational axis.

Torque, cont

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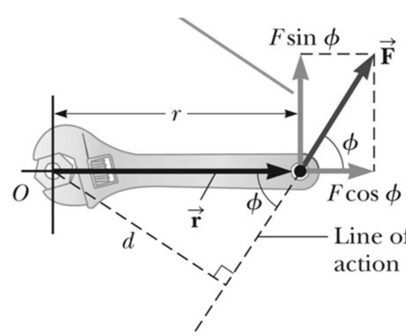
•The moment arm, d, is the *perpendicular* distance from the axis of rotation to a line drawn along the direction of the force.

$$d = rsin\varphi$$

•The horizontal component of the force ($Fcos\varphi$) has no tendency to produce a rotation.



Net Torque

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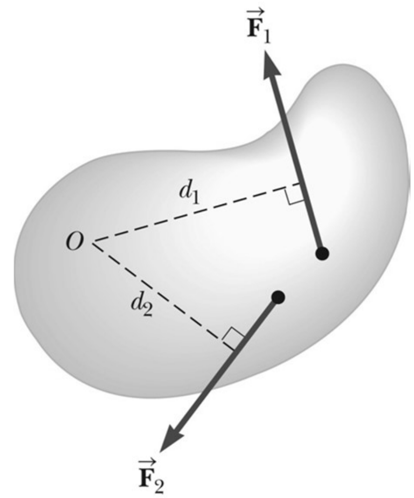
- Torque will have direction.
  - If the turning tendency of the force is counterclockwise, the torque will be positive.
  - If the turning tendency is clockwise, the torque will be negative.
- force $\vec{F}_1$ will tend to cause a counterclockwise rotation about O.
- The force $\vec{F}_2$ will tend to cause a clockwise rotation about O.

$$\sum \tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$$



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Torque vs. Force

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- Forces can cause a change in translational motion.
  - Described by Newton's Second Law
- Forces can cause a change in rotational motion.
  - The effectiveness of this change depends on the force and the moment arm.
  - The change in rotational motion depends on the torque.

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Torque Units

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- The SI units of torque are $N.m$.
  - Although torque is a force multiplied by a distance, it is very different from work and energy.
  - The units for torque are reported in $N.m$ and not changed to Joules.

Torque and Angular Acceleration

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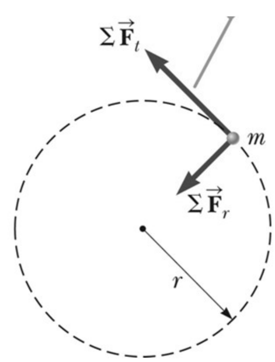
- Consider a particle of mass m rotating in a circle of radius r under the influence of tangential force .
- The tangential force provides a tangential acceleration:

$$F_t = ma_t$$

- The radial force causes the particle to move in a circular path.



Torque and Angular Acceleration, Particle cont.

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- The magnitude of the torque produced by $\sum \vec{F}_t$ on a particle about an axis through the center of the circle is

$$\sum \tau = \sum F_t r = (ma_t)r$$

- The tangential acceleration is related to the angular acceleration.

$$\sum \tau = (ma_t)r = (mr\alpha)r = mr^2\alpha$$

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Torque and Angular Acceleration, Particle cont.

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- Since mr^2 is the moment of inertia of the particle,

$$\sum \tau = I\alpha$$

- The torque is directly proportional to the angular acceleration and the constant of proportionality is the moment of inertia.

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Torque and Angular Acceleration, Extended

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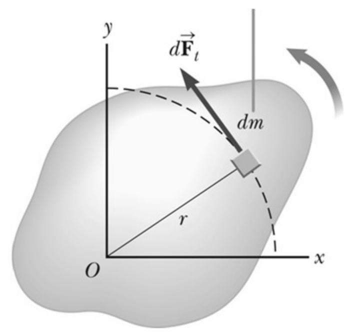
- Consider the object consists of an infinite number of mass elements *dm* of infinitesimal size.
- Each mass element rotates in a circle about the origin, *O*.
- Each mass element has a tangential acceleration.
- From Newton's Second Law
- $dF_t = (dm)a_t$
- The torque associated with the force and using the angular acceleration gives
- $d\tau_{ext} = rdF_t = a_t r dm = \alpha r^2 dm$



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Torque and Angular Acceleration, Extended cont.

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Finding the net torque: $\sum \tau_{ext} = \alpha \int r^2 dm$

This becomes

$$\sum \tau_{ext} = I\alpha$$

- This is the same relationship that applied to a particle.
- This is the mathematic representation of the analysis model of a rigid body under a net torque.
- The result also applies when the forces have radial components.
  - The line of action of the radial component must pass through the axis of rotation.
  - These components will produce zero torque about the axis.

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Work in Rotational Motion

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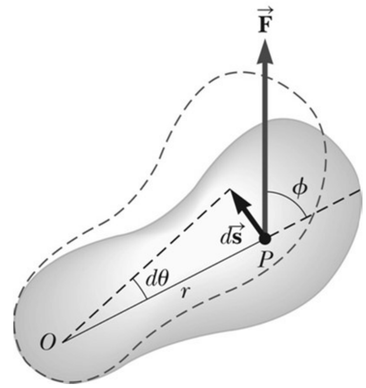
Find the work done by $\vec{F}$ on the object as it rotates through an infinitesimal distance $ds = rd\theta$.

$$dW = \vec{F} \cdot d\vec{s} = (Fsin\varphi)rd\theta$$

The radial component of the force does no work because it is perpendicular to the displacement.

$$W = rFsin\varphi \int d\theta$$

$$W = \tau\theta$$



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Power in Rotational Motion

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The rate at which work is being done in a time interval dt is

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

- This is analogous to $P = Fv$ in a linear system.

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Work-Kinetic Energy Theorem in Rotational Motion

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- The work-kinetic energy theorem for rotational motion states that:

"The net work done by external forces in rotating a symmetrical rigid object about a fixed axis equals the change in the object's rotational kinetic energy".

$$\sum W = \int_{\omega_i}^{\omega_f} I\omega \, d\omega = \frac{1}{2} I\left(\omega_f^2 - \omega_i^2\right)$$

- The rotational form can be combined with the linear form which indicates the net work done by external forces on an object is the change in its total kinetic energy, which is the sum of the translational and rotational kinetic energies.

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Rotational Motion About a Fixed Axis	**Translational Motion**
Angular speed $\omega = d\theta/dt$	Translational speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Translational acceleration $a = dv/dt$
Net torque $\Sigma\tau_{\text{ext}} = I\alpha$	Net force $\Sigma F = ma$
If $\alpha =$ constant $\begin{cases} \omega_f = \omega_i + \alpha t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If $a =$ constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$	Work $W = \int_{x_i}^{x_f} F_x \, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $P = \tau\omega$	Power $P = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

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Rolling Object

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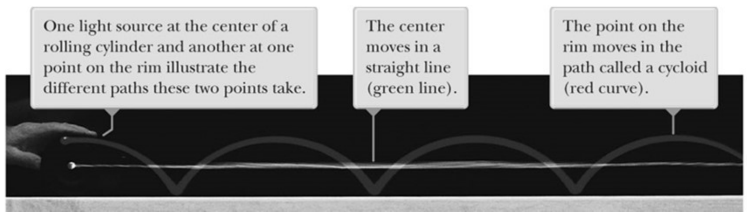
The red curve shows the path moved by a point on the rim of the object.
This path is called a cycloid.

The green line shows the path of the center of mass of the object.

In pure rolling motion, an object rolls without slipping.

In such a case, there is a simple relationship between its rotational and translational motions.



Pure Rolling Motion, Object's Center of Mass

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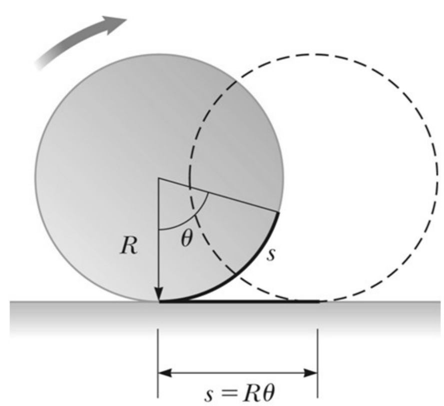
The translational speed of the center of mass is

$$v_{com} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

The linear acceleration of the center of mass is

$$a_{com} = \frac{dv_{com}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



Rolling Motion Cont.

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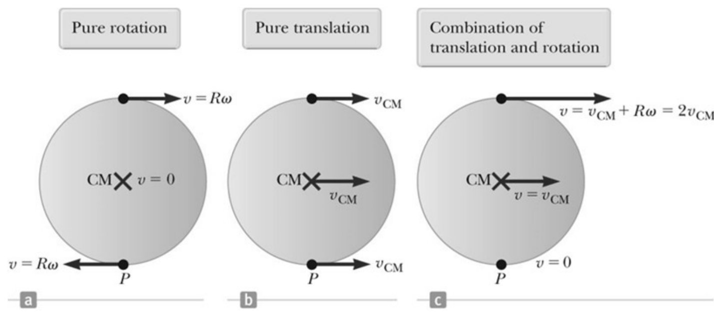
•Rolling motion can be modeled as a combination of pure translational motion and pure rotational motion.

•The contact point between the surface and the cylinder has a translational speed of zero (c).



Total Kinetic Energy of a Rolling Object

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The total kinetic energy of a rolling object is the sum of the translational energy of its center of mass and the rotational kinetic energy about its center of mass.

$$K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} M v_{com}^2$$

- The $\frac{1}{2} I_{com}\omega^2$ represents the rotational kinetic energy of the cylinder about its center of mass.
- The $\frac{1}{2} M v_{com}^2$ represents the translational kinetic energy of the cylinder about its center of mass.

Total Kinetic Energy, Example

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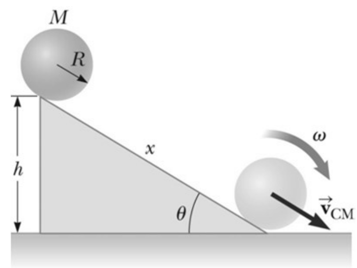
Accelerated rolling motion is possible only if friction is present between the sphere and the incline.

- The friction produces the net torque required for rotation.
- No loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant.
- In reality, rolling friction causes mechanical energy to transform to internal energy.
- Rolling friction is due to deformations of the surface and the rolling object.



Total Kinetic Energy, Example cont.

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Apply Conservation of Mechanical Energy:

- Let $U = 0$ at the bottom of the plane
- $K_f + U_f = K_i + U_i$
- $K_f = \frac{1}{2}\frac{I_{com}}{R^2}v_{com}^2 + \frac{1}{2}Mv_{com}^2 = \frac{1}{2}\left(\frac{I_{com}}{R^2} + M\right)v_{com}^2$
- $U_i = Mgh$
- $U_f = K_i = 0$

Solving for v

$$v_{com} = \sqrt{\frac{2gh}{1 + \frac{I_{com}}{MR^2}}}$$