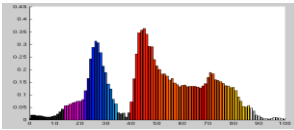


Signals and Systems

Lecture 17: Properties of Fourier Transform

Outline

- **Properties of Fourier Transform.**
- **Fourier transform properties (Table 1).**
- **Basic Fourier transform pairs (Table 2).**

Properties of Fourier Transform

The Fourier Transform possesses the following properties:

1) **Linearity.**
2) **Time shifting.**
3) **Conjugation and Conjugation symmetry.**
4) **Differentiation.**
5) **Integration.**
6) **Time scaling and time reversal.**
7) **Frequency shifting.**
8) **Duality.**
9) **Time Convolution.**
10) **Parseval's Theorem.**
11) **Modulation.**

Linearity

If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega) \text{ and } y(t) \overset{FT}{\leftrightarrow} Y(\omega)$$

Then

$$z(t) = a\,x(t) + by(t) \overset{FT}{\leftrightarrow} Z(\omega) = a\,X(\omega) + b\,Y(\omega)$$

- **Meaning: The FT of linear combination of the signals is equal to linear combination of their Fourier transforms.**

Time shifting

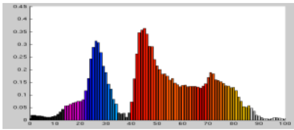
If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then

$$y(t) = x(t - t_0) \overset{FT}{\leftrightarrow} Y(\omega) = e^{-j\omega t_0} . X(\omega)$$

- **Meaning: A shift of 't_0' in time domain is equivalent to introducing a phase shift of $-\omega t_0$. But amplitude remains same.**



Conjugation and Conjugation Symmetry

If

$$x(t) \overset{FT}{\leftrightarrow} X(j\omega)$$

Then

$$x^*(t) \overset{FT}{\leftrightarrow} X^*(-j\omega)$$

Remark:

If

$x(t)$ **is real :** $x^*(t) = x(t)$

Then

$$X^*(-\omega) = X(\omega)$$

Also

$$X(-\omega) = X^*(\omega)$$

Differentiation

A. Differentiation in time:

If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then

$$d\,\frac{x(t)}{dt} \overset{FT}{\leftrightarrow} j\omega\, X(\omega)$$

- **Meaning: Differentiation in time domain corresponds to multiplying by $j\omega$ in frequency domain.**

B. Frequency Differentiation:

If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then

$$-jt\, x(t) \overset{FT}{\leftrightarrow} \frac{d}{d\omega} X(\omega)$$

- **Meaning: Differentiating the frequency spectrum is equivalent to multiplying the time domain signal by complex number $-jt$.**

Time Integration

If

$$F[x(t)] = X(\omega)$$

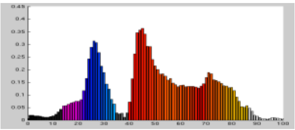
Then

$$F\left[\int_{-\infty}^{t} x(\tau).\,d\tau\right] = \frac{1}{j\omega} X(\omega) + \pi\, X(0)\, \delta(\omega)$$

Where $X(0)$ - is the intial condition.

If $X(0) = 0$ then

$$\int_{-\infty}^{t} x(\tau).\,d\tau \overset{FT}{\Leftrightarrow} \frac{1}{j\omega} X(\omega)$$



Time scaling and time reversal

If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then

$$y(t) = x(at) \overset{FT}{\leftrightarrow} Y(\omega) = \frac{1}{|a|} X(\frac{\omega}{a})$$

- **Meaning: Compression of a signal in time domain is equivalent to expansion in frequency domain and vice versa.**

For time reversal:

$$x(-t) \overset{FT}{\leftrightarrow} X(-\omega) \ \ ; \ \ put\ a = -1$$

Frequency shifting

If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then

$$e^{j\omega_0 t} . x(t) \overset{FT}{\leftrightarrow} X(\omega - \omega_0)$$

- **Meaning: Shifting the frequency by ω_0 in frequency domain is equivalent to multiplying the time domain signal by $e^{j\omega_0 t}$**

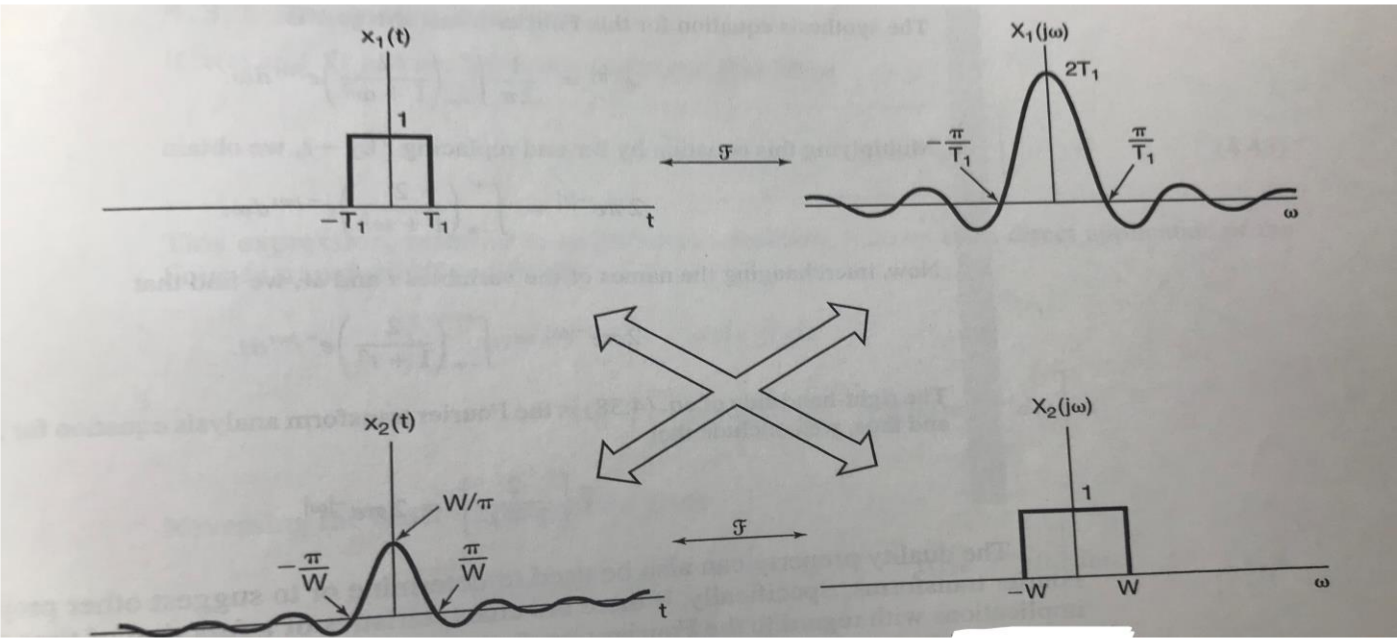
Duality

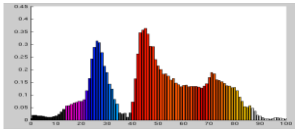
If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then

$$X(t) \overset{FT}{\leftrightarrow} 2\pi x(-\omega)$$





Time Convolution

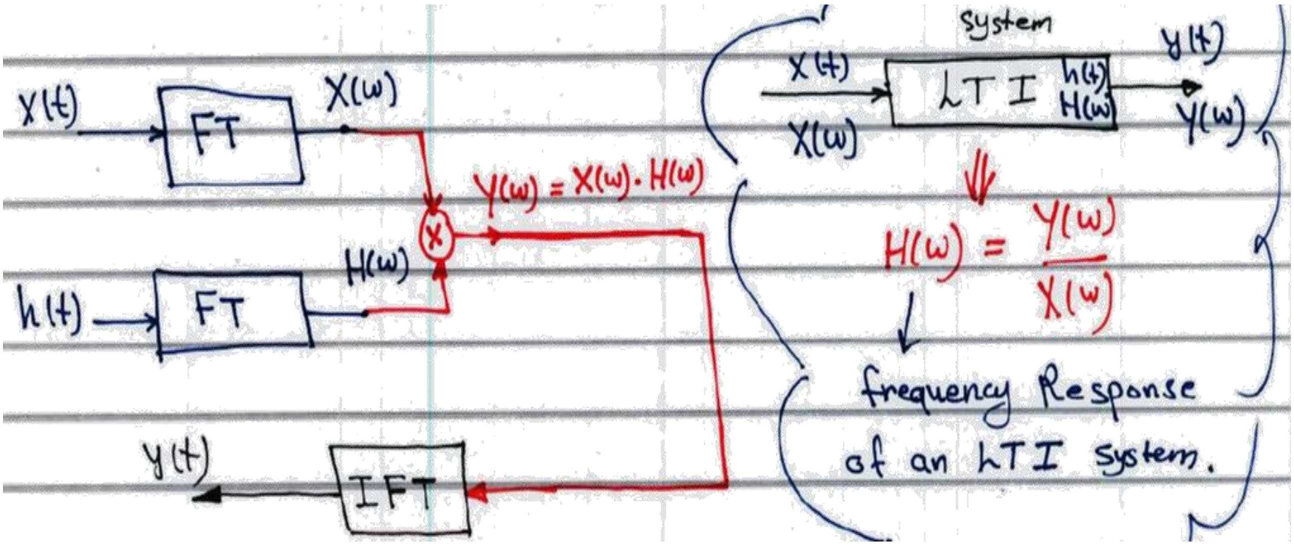
If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega) \text{ and } h(t) \overset{FT}{\leftrightarrow} H(\omega)$$

Then

$$y(t) = x(t) * h(t) \overset{FT}{\leftrightarrow} Y(\omega) = X(\omega).H(\omega)$$

- **Meaning:** A convolution operation is transformed to modulation (multiplication) in frequency domain.



Parseval's Theorem

If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}.\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Where E is the energy of the signal.

- **Meaning:** Energy of the signal can be obtained by interchanging its energy spectrum.

Modulation

If

$$x(t) \overset{FT}{\leftrightarrow} X(\omega) \text{ and } y(t) \overset{FT}{\leftrightarrow} Y(\omega)$$

Then

$$z(t) = x(t).y(t) \overset{FT}{\leftrightarrow} Z(\omega) = \frac{1}{2\pi}.[X(\omega) * Y(\omega)]$$

- **Meaning:** Modulation in time domain corresponds to convolution of spectrums in frequency domain.

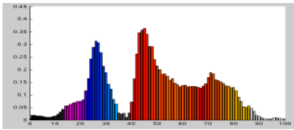


Table 1: Fourier transform properties

	Property	Time domain $x(t)$	Fourier transform $X(j\omega)$
1)	Linearity	$x(t) = A\,x_1(t) + B x_2(t)$	$X(j\omega) = A\,X_1(j\omega) + B\,X_2(j\omega)$
2)	Time shifting	$x(t - t_0)$	$e^{-j\omega t_0}\,X(j\omega)$
3)	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4)	Differentiation in time	$\frac{d^n\,x(t)}{dt^n}$	$(j\omega)^n . X(j\omega)$
5)	Differentiation in frequency	$-jt\,x(t)$	$\frac{d\,X(j\omega)}{d\omega}$
6)	Time Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi . X(0) . \delta(\omega)$
7)	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(j\frac{\omega}{a}\right)$
8)	Time reversal	$x(-t)$	$X(-j\omega)$
9)	Frequency shifting	$x(t) . e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$
10)	Duality	$X(t)$	$2\pi x(-j\omega)$
11)	Time convolution	$x(t) * h(t)$	$X(j\omega) . H(j\omega)$
12)	Parseval's Theorem	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 dt$
13)	Modulation	$z(t) = x(t) . y(t)$	$Z(\omega) = \frac{1}{2\pi} . X(j\omega) * Y(j\omega)$

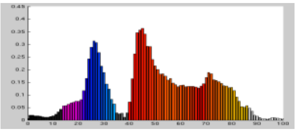


Table2: Basic Fourier transform pairs

	Signal	Fourier transform
1)	$\delta(t)$	1
2)	$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
3)	$\delta(t - t_0)$	$e^{-j\omega t_0}$
4)	$t.e^{-at}.u(t)$	$\frac{1}{(a + j\omega)^2}$
5)	$u(-t)$	$\pi\delta(\omega) - \frac{1}{j\omega}$
6)	$e^{at}.u(-t)$	$\frac{1}{a - j\omega}$
7)	$e^{-a\|t\|}$	$\frac{2a}{a^2 + \omega^2}$
8)	$cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
9)	$sin(\omega_0 t)$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
10)	$\frac{1}{a^2 + t^2}$	$e^{-a\|\omega\|}$
11)	$Sgn(t)$	$\frac{2}{j\omega}$
12)	$1; \quad for\ all\ t$	$2\pi\delta(\omega)$