Direct Methods for Solving Linear Systems

Linear Algebra & Matrix Inversion

Numerical Analysis (9th Edition)

R L Burden & J D Faires

Beamer Presentation Slides
prepared by
John Carroll
Dublin City University

Outline

Outline

Outline

Outline

Outline

Outline

Linear Algebra & Matrix Inversion: Matrices

Definition of a Matrix

An $n \times m$ (n by m) matrix is a rectangular array of elements with n rows and m columns in which not only is the value of an element important, but also its position in the array.

Linear Algebra & Matrix Inversion: Matrices

Definition of a Matrix

An $n \times m$ (n by m) matrix is a rectangular array of elements with n rows and m columns in which not only is the value of an element important, but also its position in the array.

Notation

The notation for an $n \times m$ matrix will be a capital letter such as A for the matrix and lowercase letters with double subscripts, such as a_{ij}, to refer to the entry at the intersection of the ith row and jth column; that is:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Linear Algebra & Matrix Inversion

Definition: Equality of Matrices

Two matrices A and B are equal if they have the same number of rows and columns, say $n \times m$, and if $a_{ij} = b_{ij}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Linear Algebra & Matrix Inversion

Definition: Equality of Matrices

Two matrices A and B are equal if they have the same number of rows and columns, say $n \times m$, and if $a_{ij} = b_{ij}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

This definition means, for example, that

$$\begin{bmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 7 & 0 \end{bmatrix}$$

because they differ in dimension.

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Definition: Addition of 2 Matrices

If A and B are both $n \times m$ matrices, then the sum of A and B, denoted $A + B$, is the $n \times m$ matrix whose entries are $a_{ij} + b_{ij}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Definition: Addition of 2 Matrices

If A and B are both $n \times m$ matrices, then the sum of A and B, denoted $A + B$, is the $n \times m$ matrix whose entries are $a_{ij} + b_{ij}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Definition: Scalar Multiplication

If A is an $n \times m$ matrix and λ is a real number, then the scalar multiplication of λ and A, denoted λA, is the $n \times m$ matrix whose entries are λa_{ij}, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Example

Determine $A + B$ and λA when

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & -8 \\ 0 & 1 & 6 \end{bmatrix} \quad \text{and} \quad \lambda = -2$$

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Example

Determine $A + B$ and λA when

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & -8 \\ 0 & 1 & 6 \end{bmatrix} \quad \text{and} \quad \lambda = -2$$

Solution

We have

$$A + B = \begin{bmatrix} 2+4 & -1+2 & 7-8 \\ 3+0 & 1+1 & 0+6 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -1 \\ 3 & 2 & 6 \end{bmatrix}$$

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Example

Determine $A + B$ and λA when

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & -8 \\ 0 & 1 & 6 \end{bmatrix} \qquad \text{and} \qquad \lambda = -2$$

Solution

We have

$$A + B = \begin{bmatrix} 2+4 & -1+2 & 7-8 \\ 3+0 & 1+1 & 0+6 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -1 \\ 3 & 2 & 6 \end{bmatrix}$$

and

$$\lambda A = \begin{bmatrix} -2(2) & -2(-1) & -2(7) \\ -2(3) & -2(1) & -2(0) \end{bmatrix} = \begin{bmatrix} -4 & 2 & -14 \\ -6 & -2 & 0 \end{bmatrix}$$

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers.

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

(ii) $(A + B) + C = A + (B + C)$,

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

(ii) $(A + B) + C = A + (B + C)$,

(iii) $A + O = O + A = A$,

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

(ii) $(A + B) + C = A + (B + C)$,

(iii) $A + O = O + A = A$,

(iv) $A + (-A) = -A + A = 0$,

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

(ii) $(A + B) + C = A + (B + C)$,

(iii) $A + O = O + A = A$,

(iv) $A + (-A) = -A + A = 0$,

(v) $\lambda(A + B) = \lambda A + \lambda B$,

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

(ii) $(A + B) + C = A + (B + C)$,

(iii) $A + O = O + A = A$,

(iv) $A + (-A) = -A + A = 0$,

(v) $\lambda(A + B) = \lambda A + \lambda B$,

(vi) $(\lambda + \mu)A = \lambda A + \mu A$,

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

(ii) $(A + B) + C = A + (B + C)$,

(iii) $A + O = O + A = A$,

(iv) $A + (-A) = -A + A = 0$,

(v) $\lambda(A + B) = \lambda A + \lambda B$,

(vi) $(\lambda + \mu)A = \lambda A + \mu A$,

(vii) $\lambda(\mu A) = (\lambda\mu)A$,

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

(ii) $(A + B) + C = A + (B + C)$,

(iii) $A + O = O + A = A$,

(iv) $A + (-A) = -A + A = 0$,

(v) $\lambda(A + B) = \lambda A + \lambda B$,

(vi) $(\lambda + \mu)A = \lambda A + \mu A$,

(vii) $\lambda(\mu A) = (\lambda\mu)A$,

(viii) $1A = A$.

Linear Algebra & Matrix Inversion: Matrix Arithmetic

Note: In what follows, O denotes a matrix all of whose entries are 0 and $-A$ denotes the matrix whose entries are $-a_{ij}$.

Theorem: Addition & Scalar Multiplication

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

(ii) $(A + B) + C = A + (B + C)$,

(iii) $A + O = O + A = A$,

(iv) $A + (-A) = -A + A = 0$,

(v) $\lambda(A + B) = \lambda A + \lambda B$,

(vi) $(\lambda + \mu)A = \lambda A + \mu A$,

(vii) $\lambda(\mu A) = (\lambda\mu)A$,

(viii) $1A = A$.

All these properties follow from similar results concerning the real numbers.

Outline

Linear Algebra: Matrix-Vector Products

Definition: Matrix-Vector Product

Let A be an $n \times m$ matrix and **b** an m-dimensional column vector. The matrix-vector product of A and **b**, denoted $A\mathbf{b}$, is an n-dimensional column vector given by

$$A\mathbf{b} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1i} b_i \\ \sum_{i=1}^{m} a_{2i} b_i \\ \vdots \\ \sum_{i=1}^{m} a_{ni} b_i \end{bmatrix}$$

Linear Algebra: Matrix-Vector Products

Definition: Matrix-Vector Product

Let A be an $n \times m$ matrix and $\mathbf{b}$ an m-dimensional column vector. The matrix-vector product of A and $\mathbf{b}$, denoted $A\mathbf{b}$, is an n-dimensional column vector given by

$$A\mathbf{b} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1i} b_i \\ \sum_{i=1}^{m} a_{2i} b_i \\ \vdots \\ \sum_{i=1}^{m} a_{ni} b_i \end{bmatrix}$$

Note: For this product to be defined the number of columns of the matrix A must match the number of rows of the vector $\mathbf{b}$, and the result is another column vector with the number of rows matching the number of rows in the matrix.

Linear Algebra: Matrix-Vector Products

Example

Determine the product $A\mathbf{b}$ if $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Linear Algebra: Matrix-Vector Products

Example

Determine the product $A\mathbf{b}$ if $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Solution

Because A has dimension 3×2 and $\mathbf{b}$ has dimension 2×1, the product is defined and is a vector with three rows.

Linear Algebra: Matrix-Vector Products

Example

Determine the product $A\mathbf{b}$ if $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Solution

Because A has dimension 3×2 and $\mathbf{b}$ has dimension 2×1, the product is defined and is a vector with three rows. These are

$$3(3) + 2(-1) = 7, \quad (-1)(3) + 1(-1) = -4, \quad \text{and} \quad 6(3) + 4(-1) = 14$$

Linear Algebra: Matrix-Vector Products

Example

Determine the product $A\mathbf{b}$ if $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Solution

Because A has dimension 3×2 and $\mathbf{b}$ has dimension 2×1, the product is defined and is a vector with three rows. These are

$$3(3) + 2(-1) = 7, \quad (-1)(3) + 1(-1) = -4, \quad \text{and} \quad 6(3) + 4(-1) = 14$$

That is,

$$A\mathbf{b} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 14 \end{bmatrix}$$

Linear Algebra: Matrix-Vector Products

Describing a Linear System

The introduction of the matrix-vector product permits us to view the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots \qquad\qquad & \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

as the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

Linear Algebra: Matrix-Vector Products

Describing a Linear System (Cont'd)

- where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

because all the entries in the product $A\mathbf{x}$ must match the corresponding entries in the vector $\mathbf{b}$.

Linear Algebra: Matrix-Vector Products

Describing a Linear System (Cont'd)

- where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

  because all the entries in the product $A\mathbf{x}$ must match the corresponding entries in the vector $\mathbf{b}$.

- In essence, then, an $n \times m$ matrix is a function with domain the set of m-dimensional column vectors and range a subset of the n-dimensional column vectors.

Linear Algebra: Matrix-Matrix Products

We can use matrix-vector multiplication to define general matrix-matrix multiplication.

Linear Algebra: Matrix-Matrix Products

We can use matrix-vector multiplication to define general matrix-matrix multiplication.

Definition: Matrix-Matrix Product

Let A be an $n \times m$ matrix and B an $m \times p$ matrix. The matrix product of A and B, denoted AB, is an $n \times p$ matrix C whose entries c_{ij} are

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj},$$

for each $i = 1, 2, \ldots n$, and $j = 1, 2, \ldots, p$.

Linear Algebra: Matrix-Matrix Products

Row by Column Multiplication

The computation of c_{ij} can be viewed as the multiplication of the entries of the ith row of A with corresponding entries in the jth column of B, followed by a summation; that is,

$$[a_{i1}, a_{i2}, \ldots, a_{im}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = c_{ij}$$

Linear Algebra: Matrix-Matrix Products

Row by Column Multiplication

The computation of c_{ij} can be viewed as the multiplication of the entries of the ith row of A with corresponding entries in the jth column of B, followed by a summation; that is,

$$[a_{i1}, a_{i2}, \ldots, a_{im}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = c_{ij}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj}$$

Linear Algebra: Matrix-Matrix Products

Row by Column Multiplication

The computation of c_{ij} can be viewed as the multiplication of the entries of the ith row of A with corresponding entries in the jth column of B, followed by a summation; that is,

$$[a_{i1}, a_{i2}, \ldots, a_{im}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = c_{ij}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj}$$

This explains why the number of columns of A must equal the number of rows of B for the product AB to be defined.

Linear Algebra: Matrix-Matrix Products

Example

Determine all possible products of the matrices

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & 0 & 1 \\ -1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

Linear Algebra: Matrix-Matrix Products

Solution (1/2)

The size of the matrices are

A	B	C	D
3×2	2×3	3×4	2×2

Linear Algebra: Matrix-Matrix Products

Solution (1/2)

The size of the matrices are

A	B	C	D
3×2	2×3	3×4	2×2

The products that can be defined, and their dimensions, are:

AB	BA	AD	BC	DB	DD
3×3	2×2	3×2	2×4	2×3	2×2

Linear Algebra: Matrix-Matrix Products

Solution (2/2)

These products are

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix}$$

Linear Algebra: Matrix-Matrix Products

Solution (2/2)

These products are

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}$$

Linear Algebra: Matrix-Matrix Products

Solution (2/2)

These products are

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}$$

$$AD = \begin{bmatrix} 7 & -5 \\ 1 & 0 \\ 9 & -5 \end{bmatrix}$$

Linear Algebra: Matrix-Matrix Products

Solution (2/2)

These products are

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}$$

$$AD = \begin{bmatrix} 7 & -5 \\ 1 & 0 \\ 9 & -5 \end{bmatrix} \qquad BC = \begin{bmatrix} 2 & 4 & 0 & 3 \\ 7 & 8 & 6 & 4 \end{bmatrix}$$

Linear Algebra: Matrix-Matrix Products

Solution (2/2)

These products are

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}$$

$$AD = \begin{bmatrix} 7 & -5 \\ 1 & 0 \\ 9 & -5 \end{bmatrix} \qquad BC = \begin{bmatrix} 2 & 4 & 0 & 3 \\ 7 & 8 & 6 & 4 \end{bmatrix}$$

$$DB = \begin{bmatrix} -1 & 0 & -3 \\ 1 & 1 & -4 \end{bmatrix}$$

Linear Algebra: Matrix-Matrix Products

Solution (2/2)

These products are

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}$$

$$AD = \begin{bmatrix} 7 & -5 \\ 1 & 0 \\ 9 & -5 \end{bmatrix} \qquad BC = \begin{bmatrix} 2 & 4 & 0 & 3 \\ 7 & 8 & 6 & 4 \end{bmatrix}$$

$$DB = \begin{bmatrix} -1 & 0 & -3 \\ 1 & 1 & -4 \end{bmatrix} \qquad DD = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Linear Algebra: Matrix-Matrix Products

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}$$

Non-Commutativity

Linear Algebra: Matrix-Matrix Products

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}$$

Non-Commutativity

- Notice that although the matrix products AB and BA are both defined, their results are very different; they do not even have the same dimension.

Linear Algebra: Matrix-Matrix Products

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}$$

Non-Commutativity

- Notice that although the matrix products AB and BA are both defined, their results are very different; they do not even have the same dimension.
- In mathematical language, we say that the matrix product operation is not commutative, that is, products in reverse order can differ.

Linear Algebra: Matrix-Matrix Products

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}$$

Non-Commutativity

- Notice that although the matrix products AB and BA are both defined, their results are very different; they do not even have the same dimension.
- In mathematical language, we say that the matrix product operation is not commutative, that is, products in reverse order can differ.
- This is the case even when both products are defined and are of the same dimension. Almost any example will show this.

Linear Algebra: Matrix-Matrix Products

Certain important operations involving matrix product do hold, however, as indicated in the following result.

Linear Algebra: Matrix-Matrix Products

Certain important operations involving matrix product do hold, however, as indicated in the following result.

Theorem

Let A be an $n \times m$ matrix, B be an $m \times k$ matrix, C be a $k \times p$ matrix, D be an $m \times k$ matrix, and λ be a real number. The following properties hold:

(a) $A(BC) = (AB)C$

(b) $A(B + D) = AB + AD$

(c) $\lambda(AB) = (\lambda A)B = A(\lambda B)$

Linear Algebra: Matrix-Matrix Products

Certain important operations involving matrix product do hold, however, as indicated in the following result.

Theorem

Let A be an $n \times m$ matrix, B be an $m \times k$ matrix, C be a $k \times p$ matrix, D be an $m \times k$ matrix, and λ be a real number. The following properties hold:

(a) $A(BC) = (AB)C$

(b) $A(B + D) = AB + AD$

(c) $\lambda(AB) = (\lambda A)B = A(\lambda B)$

The verification of the property in part (a) will only be presented here in order to show the method involved. The other parts can be shown in a similar manner.

Linear Algebra: Matrix-Matrix Products

$A(BC) = (AB)C$: Proof (1/3)

Linear Algebra: Matrix-Matrix Products

$A(BC) = (AB)C$: Proof (1/3)

To show that $A(BC) = (AB)C$, compute the ij-entry of each side of the equation. BC is an $m \times p$ matrix with ij-entry

$$(BC)_{sj} = \sum_{l=1}^{k} b_{sl} c_{lj}$$

Linear Algebra: Matrix-Matrix Products

$A(BC) = (AB)C$: Proof (1/3)

To show that $A(BC) = (AB)C$, compute the ij-entry of each side of the equation. BC is an $m \times p$ matrix with ij-entry

$$(BC)_{sj} = \sum_{l=1}^{k} b_{sl} c_{lj}$$

Thus, $A(BC)$ is an $n \times p$ matrix with entries

$$[A(BC)]_{ij} = \sum_{s=1}^{m} a_{is} (BC)_{sj} = \sum_{s=1}^{m} a_{is} \left(\sum_{l=1}^{k} b_{sl} c_{lj} \right) = \sum_{s=1}^{m} \sum_{l=1}^{k} a_{is} b_{sl} c_{lj}$$

Linear Algebra: Matrix-Matrix Products

$A(BC) = (AB)C$: Proof (2/3)

Linear Algebra: Matrix-Matrix Products

$A(BC) = (AB)C$: Proof (2/3)

Similarly, AB is an $n \times k$ matrix with entries

$$(AB)_{il} = \sum_{s=1}^{m} a_{is} b_{sl}$$

Linear Algebra: Matrix-Matrix Products

$A(BC) = (AB)C$: Proof (2/3)

Similarly, AB is an $n \times k$ matrix with entries

$$(AB)_{il} = \sum_{s=1}^{m} a_{is} b_{sl}$$

so $(AB)C$ is an $n \times p$ matrix with entries

$$[(AB)C]_{ij} = \sum_{l=1}^{k} (AB)_{il} c_{lj} = \sum_{l=1}^{k} \left(\sum_{s=1}^{m} a_{is} b_{sl} \right) c_{lj} = \sum_{l=1}^{k} \sum_{s=1}^{m} a_{is} b_{sl} c_{lj}$$

Linear Algebra: Matrix-Matrix Products

$$[(AB)C]_{ij} = \sum_{l=1}^{k} (AB)_{il} c_{lj} = \sum_{l=1}^{k} \left(\sum_{s=1}^{m} a_{is} b_{sl} \right) c_{lj} = \sum_{l=1}^{k} \sum_{s=1}^{m} a_{is} b_{sl} c_{lj}$$

$A(BC) = (AB)C$: Proof (3/3)

Linear Algebra: Matrix-Matrix Products

$$[(AB)C]_{ij} = \sum_{l=1}^{k} (AB)_{il} c_{lj} = \sum_{l=1}^{k} \left(\sum_{s=1}^{m} a_{is} b_{sl} \right) c_{lj} = \sum_{l=1}^{k} \sum_{s=1}^{m} a_{is} b_{sl} c_{lj}$$

$A(BC) = (AB)C$: Proof (3/3)

Interchanging the order of summation on the right side gives

$$[(AB)C]_{ij} = \sum_{s=1}^{m} \sum_{l=1}^{k} a_{is} b_{sl} c_{lj} = [A(BC)]_{ij}$$

for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, p$. So $A(BC) = (AB)C$.

Outline

Linear Algebra: Square Matrices

Definition: Square, Diagonal & Idenity Matrices

(i) A square matrix has the same number of rows as columns.

Linear Algebra: Square Matrices

Definition: Square, Diagonal & Idenity Matrices

(i) A square matrix has the same number of rows as columns.

(ii) A diagonal matrix $D = [d_{ij}]$ is a square matrix with $d_{ij} = 0$ whenever $i \neq j$.

Linear Algebra: Square Matrices

Definition: Square, Diagonal & Idenity Matrices

(i) A square matrix has the same number of rows as columns.

(ii) A diagonal matrix $D = [d_{ij}]$ is a square matrix with $d_{ij} = 0$ whenever $i \neq j$.

(iii) The identity matrix of order n, $I_n = [\delta_{ij}]$, is a diagonal matrix whose diagonal entries are all 1s. When the size of I_n is clear, this matrix is generally written simply as I.

Linear Algebra: Square Matrices

Definition: Square, Diagonal & Idenity Matrices

(i) A square matrix has the same number of rows as columns.

(ii) A diagonal matrix $D = [d_{ij}]$ is a square matrix with $d_{ij} = 0$ whenever $i \neq j$.

(iii) The identity matrix of order n, $I_n = [\delta_{ij}]$, is a diagonal matrix whose diagonal entries are all 1s. When the size of I_n is clear, this matrix is generally written simply as I.

For example, a diagonal matrix D of order 3 and an identity matrix I of order 3 are:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Linear Algebra: Square Matrices

Definition: Upper- & Lower-Triangular Matrices

An upper-triangular $n \times n$ matrix $U = [u_{ij}]$ has, for each $j = 1, 2, \ldots, n$, the entries

$$u_{ij} = 0, \quad \text{for each } i = j+1, j+2, \ldots, n$$

Linear Algebra: Square Matrices

Definition: Upper- & Lower-Triangular Matrices

An upper-triangular $n \times n$ matrix $U = [u_{ij}]$ has, for each $j = 1, 2, \ldots, n$, the entries

$$u_{ij} = 0, \quad \text{for each } i = j+1, j+2, \ldots, n$$

and a lower-triangular matrix $L = [l_{ij}]$ has, for each $j = 1, 2, \ldots, n$, the entries

$$l_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, j-1$$

Linear Algebra: Square Matrices

Definition: Upper- & Lower-Triangular Matrices

An upper-triangular $n \times n$ matrix $U = [u_{ij}]$ has, for each $j = 1, 2, \ldots, n$, the entries

$$u_{ij} = 0, \quad \text{for each } i = j+1, j+2, \ldots, n$$

and a lower-triangular matrix $L = [l_{ij}]$ has, for each $j = 1, 2, \ldots, n$, the entries

$$l_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, j-1$$

A diagonal matrix, then, is both both upper triangular and lower triangular because its only nonzero entries must lie on the main diagonal.

Linear Algebra: Square Matrices

Example

Consider the identity matrix of order three,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Linear Algebra: Square Matrices

Example

Consider the identity matrix of order three,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is any 3×3 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Linear Algebra: Square Matrices

Example

Consider the identity matrix of order three,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is any 3×3 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Linear Algebra: Square Matrices

Example

Consider the identity matrix of order three,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is any 3×3 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

Linear Algebra: Square Matrices

Example

Consider the identity matrix of order three,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is any 3×3 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

The identity matrix I_n commutes with any $n \times n$ matrix A; that is, the order of multiplication does not matter, $I_nA = A = AI_n$. Keep in mind that this property is not true in general, even for square matrices.

Outline

Linear Algebra: Inverse Matrices

Definition: Matrix Invesre

An $n \times n$ matrix A is said to be nonsingular (or invertible) if an $n \times n$ matrix A^{-1} exists with $AA^{-1} = A^{-1}A = I$. The matrix A^{-1} is called the inverse of A. A matrix without an inverse is called singular (or noninvertible).

Linear Algebra: Inverse Matrices

Definition: Matrix Invesre

An $n \times n$ matrix A is said to be nonsingular (or invertible) if an $n \times n$ matrix A^{-1} exists with $AA^{-1} = A^{-1}A = I$. The matrix A^{-1} is called the inverse of A. A matrix without an inverse is called singular (or noninvertible).

Theorem: Properties of the Matrix Invere

For any nonsingular $n \times n$ matrix A:

(i) A^{-1} is unique

(ii) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$

(iii) If B is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$

Linear Algebra: Inverse Matrices

Example

Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Linear Algebra: Inverse Matrices

Example

Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Show that $B = A^{-1}$,

Linear Algebra: Inverse Matrices

Example

Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Show that $B = A^{-1}$, and that the solution to the linear system described by

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2, \\ 2x_1 + x_2 \phantom{{}+2x_3} &= 3, \\ -x_1 + x_2 + 2x_3 &= 4 \end{aligned}$$

is given by the entries in $B\mathbf{b}$, where $\mathbf{b}$ is the column vector with entries 2, 3, and 4.

Linear Algebra: Inverse Matrices

Solution (1/3)

First note that

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Linear Algebra: Inverse Matrices

Solution (1/3)

First note that

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Linear Algebra: Inverse Matrices

Solution (1/3)

First note that

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Linear Algebra: Inverse Matrices

Solution (1/3)

First note that

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

In a similar manner, $BA = I_3$, so A and B are both nonsingular with $B = A^{-1}$ and $A = B^{-1}$.

Linear Algebra: Inverse Matrices

Solution (2/3)

Now convert the given linear system to the matrix equation

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Linear Algebra: Inverse Matrices

Solution (2/3)

Now convert the given linear system to the matrix equation

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

and multiply both sides by B, the inverse of A. Because we have both

$$B(A\mathbf{x}) = (BA)\mathbf{x} = I_3\mathbf{x} = \mathbf{x} \qquad \text{and} \qquad B(A\mathbf{x}) = \mathbf{b}$$

Linear Algebra: Inverse Matrices

Solution (3/3)

we have

$$BA\mathbf{x} = \left(\begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{3}{9} & \frac{3}{9} & \frac{3}{9} \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \right) \mathbf{x} = \mathbf{x}$$

Linear Algebra: Inverse Matrices

Solution (3/3)

we have

$$BA\,\mathbf{x} = \left(\begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{3}{9} & \frac{3}{9} & \frac{3}{9} \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \right) \mathbf{x} = \mathbf{x}$$

and

$$BA\,\mathbf{x} = B\,(\mathbf{b}) = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Linear Algebra: Inverse Matrices

Solution (3/3)

we have

$$BA\,\mathbf{x} = \left(\begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{3}{9} & \frac{3}{9} & \frac{3}{9} \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \right) \mathbf{x} = \mathbf{x}$$

and

$$BA\,\mathbf{x} = B\,(\mathbf{b}) = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{9} \\ \frac{13}{9} \\ \frac{5}{3} \end{bmatrix}$$

Linear Algebra: Inverse Matrices

Solution (3/3)

we have

$$BA\,\mathbf{x} = \left(\begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{3}{9} & \frac{3}{9} & \frac{3}{9} \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \right) \mathbf{x} = \mathbf{x}$$

and

$$BA\,\mathbf{x} = B\,(\mathbf{b}) = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{9} \\ \frac{13}{9} \\ \frac{5}{3} \end{bmatrix}$$

This implies that $\mathbf{x} = B\mathbf{b}$ and gives the solution $x_1 = 7/9$, $x_2 = 13/9$, and $x_3 = 5/3$.

Linear Algebra: Inverse Matrices

A Method to Compute the Matrix Inverse

To find a method of computing A^{-1} assuming A is nonsingular, let us look again at matrix multiplication.

Linear Algebra: Inverse Matrices

A Method to Compute the Matrix Inverse

To find a method of computing A^{-1} assuming A is nonsingular, let us look again at matrix multiplication. Let B_j be the jth column of the $n \times n$ matrix B,

$$B_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

Linear Algebra: Inverse Matrices

A Method to Compute the Matrix Inverse (Cont'd)

If $AB = C$, then the jth column of C is given by the product

$$\begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{bmatrix} = C_j = AB_j \quad = \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

Linear Algebra: Inverse Matrices

A Method to Compute the Matrix Inverse (Cont'd)

If $AB = C$, then the jth column of C is given by the product

$$\begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{bmatrix} = C_j = AB_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} a_{1k} b_{kj} \\ \sum_{k=1}^{n} a_{2k} b_{kj} \\ \vdots \\ \sum_{k=1}^{n} a_{nk} b_{kj} \end{bmatrix}$$

Linear Algebra: Inverse Matrices

A Method to Compute the Matrix Inverse (Cont'd)

Suppose that A^{-1} exists and that $A^{-1} = B = (b_{ij})$.

Linear Algebra: Inverse Matrices

A Method to Compute the Matrix Inverse (Cont'd)

Suppose that A^{-1} exists and that $A^{-1} = B = (b_{ij})$. Then $AB = I$ and

$$AB_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \text{where the value 1 appears in the } j\text{th row}$$

Linear Algebra: Inverse Matrices

A Method to Compute the Matrix Inverse (Cont'd)

Suppose that A^{-1} exists and that $A^{-1} = B = (b_{ij})$. Then $AB = I$ and

$$AB_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \text{where the value 1 appears in the } j\text{th row}$$

To find B we need to solve n linear systems in which the jth column of the inverse is the solution of the linear system with right-hand side the jth column of I.

Linear Algebra: Inverse Matrices

Example

Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

Linear Algebra: Inverse Matrices

Solution (1/5)

We first consider the product AB, where B is an arbitrary 3×3 matrix.

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Linear Algebra: Inverse Matrices

Solution (1/5)

We first consider the product AB, where B is an arbitrary 3×3 matrix.

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} + 2b_{21} - b_{31} & b_{12} + 2b_{22} - b_{32} & b_{13} + 2b_{23} - b_{33} \\ 2b_{11} + b_{21} & 2b_{12} + b_{22} & 2b_{13} + b_{23} \\ -b_{11} + b_{21} + 2b_{31} & -b_{12} + b_{22} + 2b_{32} & -b_{13} + b_{23} + 2b_{33} \end{bmatrix}$$

Linear Algebra: Inverse Matrices

Solution (1/5)

We first consider the product AB, where B is an arbitrary 3×3 matrix.

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} + 2b_{21} - b_{31} & b_{12} + 2b_{22} - b_{32} & b_{13} + 2b_{23} - b_{33} \\ 2b_{11} + b_{21} & 2b_{12} + b_{22} & 2b_{13} + b_{23} \\ -b_{11} + b_{21} + 2b_{31} & -b_{12} + b_{22} + 2b_{32} & -b_{13} + b_{23} + 2b_{33} \end{bmatrix}$$

If $B = A^{-1}$, then $AB = I$.

Linear Algebra: Inverse Matrices

Solution (2/5)

If $B = A^{-1}$, then $AB = I$.

Linear Algebra: Inverse Matrices

Solution (2/5)

If $B = A^{-1}$, then $AB = I$. Therefore:

$$\begin{aligned}
b_{11} + 2b_{21} - b_{31} &= 1 \\
b_{12} + 2b_{22} - b_{32} &= 0 \\
b_{13} + 2b_{23} - b_{33} &= 0 \\
2b_{11} + b_{21} &= 0 \\
2b_{12} + b_{22} &= 1 \\
2b_{13} + b_{23} &= 0 \\
-b_{11} + b_{21} + 2b_{31} &= 0 \\
-b_{12} + b_{22} + 2b_{32} &= 0 \\
-b_{13} + b_{23} + 2b_{33} &= 1
\end{aligned}$$

Linear Algebra: Inverse Matrices

Solution (3/5)

- Notice that the coefficients in each of the systems of equations are the same, the only change in the systems occurs on the right side of the equations.

Linear Algebra: Inverse Matrices

Solution (3/5)

- Notice that the coefficients in each of the systems of equations are the same, the only change in the systems occurs on the right side of the equations.
- As a consequence, Gaussian elimination can be performed on a larger augmented matrix formed by combining the matrices for each of the systems:

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Linear Algebra: Inverse Matrices

Solution (4/5)

First, performing $(E_2 - 2E_1) \rightarrow (E_2)$

Linear Algebra: Inverse Matrices

Solution (4/5)

First, performing $(E_2 - 2E_1) \rightarrow (E_2)$ and $(E_3 + E_1) \rightarrow (E_3)$,

Linear Algebra: Inverse Matrices

Solution (4/5)

First, performing $(E_2 - 2E_1) \rightarrow (E_2)$ and $(E_3 + E_1) \rightarrow (E_3)$, followed by $(E_3 + E_2) \rightarrow (E_3)$

Linear Algebra: Inverse Matrices

Solution (4/5)

First, performing $(E_2 - 2E_1) \to (E_2)$ and $(E_3 + E_1) \to (E_3)$, followed by $(E_3 + E_2) \to (E_3)$ produces

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right]$$

Linear Algebra: Inverse Matrices

Solution (4/5)

First, performing $(E_2 - 2E_1) \rightarrow (E_2)$ and $(E_3 + E_1) \rightarrow (E_3)$, followed by $(E_3 + E_2) \rightarrow (E_3)$ produces

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right]$$

Backward substitution is performed on each of the three augmented matrices:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & 3 & -1 \end{array}\right] \quad \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right] \quad \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right]$$

Linear Algebra: Inverse Matrices

Solution (5/5)

Backsubstitution eventually gives

$$b_{11} = -\frac{2}{9} \qquad b_{12} = \frac{5}{9} \qquad \qquad b_{13} = -\frac{1}{9}$$
$$b_{21} = \frac{4}{9} \qquad b_{22} = -\frac{1}{9} \qquad \text{and} \qquad b_{23} = \frac{2}{9}$$
$$b_{31} = -\frac{1}{3} \qquad b_{32} = \frac{1}{3} \qquad \qquad b_{32} = \frac{1}{3}$$

Linear Algebra: Inverse Matrices

Solution (5/5)

Backsubstitution eventually gives

$$b_{11} = -\frac{2}{9} \qquad b_{12} = \frac{5}{9} \qquad b_{13} = -\frac{1}{9}$$
$$b_{21} = \frac{4}{9} \qquad b_{22} = -\frac{1}{9} \quad \text{and} \quad b_{23} = \frac{2}{9}$$
$$b_{31} = -\frac{1}{3} \qquad b_{32} = \frac{1}{3} \qquad b_{32} = \frac{1}{3}$$

As shown in an earlier example, these are the entries of A^{-1}:

$$B = A^{-1} = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Outline

Linear Algebra: Matrix Transpose

Definition: Matrix Transpose

The transpose of an $n \times m$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix $A^t = [a_{ji}]$, where for each i, the ith column of A^t is the same as the ith row of A. A square matrix A is called symmetric if $A = A^t$.

Linear Algebra: Matrix Transpose

Illustration

The matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 7 \\ 3 & -5 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Linear Algebra: Matrix Transpose

Illustration

The matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 7 \\ 3 & -5 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

have transposes

$$A^t = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 5 & 5 \\ 0 & -1 & -6 \end{bmatrix}, \quad B^t = \begin{bmatrix} 2 & 3 \\ 4 & -5 \\ 7 & -1 \end{bmatrix}, \quad C^t = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Linear Algebra: Matrix Transpose

Illustration

The matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 7 \\ 3 & -5 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

have transposes

$$A^t = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 5 & 5 \\ 0 & -1 & -6 \end{bmatrix}, \quad B^t = \begin{bmatrix} 2 & 3 \\ 4 & -5 \\ 7 & -1 \end{bmatrix}, \quad C^t = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

The matrix C is symmetric because $C^t = C$.

Linear Algebra: Matrix Transpose

Illustration

The matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 7 \\ 3 & -5 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

have transposes

$$A^t = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 5 & 5 \\ 0 & -1 & -6 \end{bmatrix}, \quad B^t = \begin{bmatrix} 2 & 3 \\ 4 & -5 \\ 7 & -1 \end{bmatrix}, \quad C^t = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

The matrix C is symmetric because $C^t = C$. The matrices A and B are not symmetric.

Linear Algebra: Matrix Transpose

The proof of the next result follows directly from the definition of the transpose.

Linear Algebra: Matrix Transpose

The proof of the next result follows directly from the definition of the transpose.

Theorem

The following operations involving the transpose of a matrix hold whenever the operation is possible:

Linear Algebra: Matrix Transpose

The proof of the next result follows directly from the definition of the transpose.

Theorem

The following operations involving the transpose of a matrix hold whenever the operation is possible:

(i) $(A^t)^t = A$

Linear Algebra: Matrix Transpose

The proof of the next result follows directly from the definition of the transpose.

Theorem

The following operations involving the transpose of a matrix hold whenever the operation is possible:

(i) $(A^t)^t = A$

(ii) $(A + B)^t = A^t + B^t$

Linear Algebra: Matrix Transpose

The proof of the next result follows directly from the definition of the transpose.

Theorem

The following operations involving the transpose of a matrix hold whenever the operation is possible:

(i) $(A^t)^t = A$

(ii) $(A + B)^t = A^t + B^t$

(iii) $(AB)^t = B^t A^t$

Linear Algebra: Matrix Transpose

The proof of the next result follows directly from the definition of the transpose.

Theorem

The following operations involving the transpose of a matrix hold whenever the operation is possible:

(i) $(A^t)^t = A$

(ii) $(A+B)^t = A^t + B^t$

(iii) $(AB)^t = B^t A^t$

(iv) if A^{-1} exists, then $(A^{-1})^t = (A^t)^{-1}$

Questions?